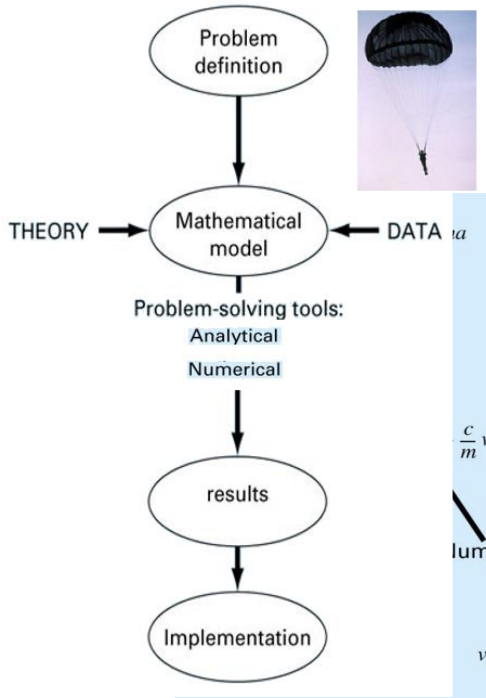
APPENDIX 1.1 MATHEMATICAL MODELING AND ENGINEERING PROBLEM SOLVING

Mathematical model plays an important role in engineering problem solving. The engineering problem solving process in parachute design is illustrated as follows:

Problem: Impact velocity upon landing is an important parameter when designing a parachute. Develop a mathematical model to solve the velocity for a falling parachutist.



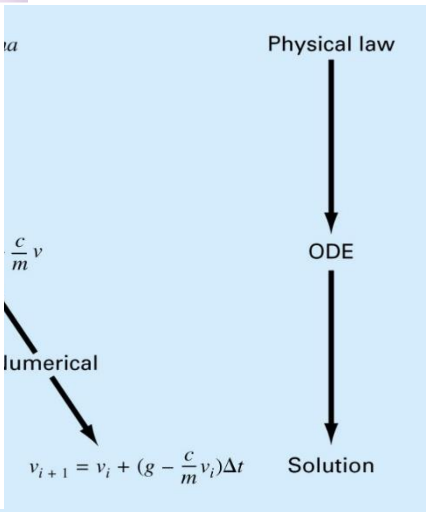


Figure A1.1 (a): Problem solving process in parachute design

To develop the mathematical model for calculating the velocity of a parachute, we can use the knowledge from the existing theory/physical law (e.g. Newton's Law) or understand the problem by empirical means (e.g. by observation and experiment). In this case, Newton's 2nd Law is applied, where $\sum F(t) = ma(t)$. A free-body diagram is drawn as below. Assume that the drag force due to air resistance is proportional to the falling velocity, the downward motion is positive, and $g = 9.81ms^{-2}$. Then, we are able to develop the mathematical model.

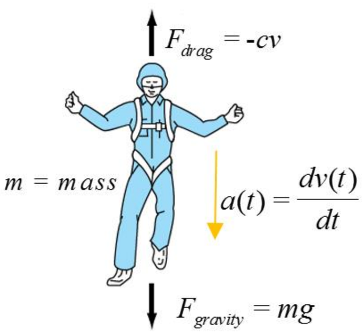


Figure A1.1 (b): Free-body diagram of the falling parachutist

The mathematical modelling of the falling parachutist's velocity:

$$\sum F(t) = ma(t) = m\frac{dv(t)}{dt}.$$

$$F_{gravity} + F_{drag} = m\frac{dv(t)}{dt}$$

$$Rate\ of\ velocity\ change = \frac{dv(t)}{dt} = \frac{F_{gravity} + F_{drag}}{m} = g - \frac{cv(t)}{m}$$

Rearrange it, we found that the mathematical model is in the form of first order linear nonhomogeneous Ordinary Differential Equation (ODE):

1[st] order ODE format: $a_1(t)v' + a_0(t)v = g(t)$

Parachutist problem: $\frac{dv(t)}{dt} + \frac{c}{m}(v(t)) = g$

Where,
First parameter, $a_1(t) = 1$,

Second parameter, $a_0(t) = \frac{c}{m}$,

Forcing function, $g(t) = g$,

Independent variable = time, (t)

Dependent variable = velocity, $v(t)$

First derivative, $' = \frac{dv}{dt}$.

The classification of the order, linear vs nonlinear, homogeneous vs nonhomogeneous will be covered later.

In general, a mathematical model can be broadly defined as a formulation or an equation that expresses the essential features of a physical system or process in mathematical terms. It can be represented as a functional relationship of the following form and explained in Table 12.1.

$$Dependent\ variable = f(independent\ variables, parameters, forcing\ functions)$$

Table A1.1: Elements of mathematical model and its example of falling parachutist problem.

Elements of mathematical model	**Example: Falling parachutist problem**
(i) <u>*Dependent variable*</u> -The characteristic that you are looking for, which reflects the behavior of the system	The falling velocity, $v(t)$ is the dependent variable that we are looking for.
(ii) <u>*Independent variable*</u> -indicates the dimension of the examined system, such as time, space	Time, (t) is the independent variable or the dimension of the problem that we are working.
(iii) <u>*Parameters*</u> -reflective of the system's properties	The parameters of this problem are the properties of the system such as the mass of parachutist & the parachute, (m) and drag coefficient of air resistance, (c).
(iv) <u>*Forcing functions*</u> -external influences acting upon the system	The acceleration due to gravity, (g) is the forcing function in this problem.

We need to solve the mathematical model (which is in differential equation such as $\frac{dv(t)}{dt} + \frac{c}{m}v(t) = g$) to obtain the velocity function, $v(t)$. To do this, two main problem-solving tools can be implemented such as the analytical method and the numerical method. In this study, we will learn how to implement the analytical method (i.e. calculus & ODE) to solve the differential equation. The desired solution obtained by analytical method-1st order ODE is given as follows: $v(t) = \frac{gm}{c}(1 - e^{-(c/m)t})$.

With this solution, it helps us to gain intuition about what to expect on the behaviour of the examined system. For example, what are the suitable mass and damping to reduce the impact velocity of the parachutist to the ground? This approach is known as the result interpretation and analysis. With an accurate mathematical model, we can predict the performance of the design without testing with the real subject. Avoiding the unnecessary physical cycles of 'modify-and-test' would save time and money.

In addition, numerical method is used to solve a complicated mathematical model that can't be solved by using analytical method. The desired solution obtained by numerical method- Euler method is given as follows: $v(t_{i+1}) = v(t_i) + [g - \frac{c}{m}v(t_i)](t_{i+1} - t_i)$. This is for your extra information and you will learn the numerical method in advanced mathematic class.

APPENDIX 1.2 CONSERVATION LAWS AND ENGINEERING

When we employed the Newton's law to develop a force balance equation for the falling parachutist, i.e. $\frac{dv(t)}{dt} = \frac{mg - cv(t)}{m}$, we can eventually boil down it to a simple equation:

(i) Transient problem:	[Change=increase-decrease] or [$\frac{dv(t)}{dt} \neq 0$]

Although simple, it embodies one of the most fundamental ways in which conservation laws are used in engineering – that is, to predict changes with respect to time. In this case, it is recognized as *time-variant (or transient) problem.* (For example, the falling velocity of the parachutist with $m = 68.1kg$, $c = 12.5kg/s$ changes with respect to time is given in Figure A1.2 (a).)

Asides from predicting changes, another way in which conservation laws are applied is for cases where change is nonexistent. In this case, it is recognized as time-invariant (or steady state) problem (For example, we would like to know when will the falling velocity becomes constant, i.e. the terminal velocity).

(ii) Steady state problem:	[Change=increase-decrease=0] or [$\frac{dv(t)}{dt} = 0$]

For falling parachutist case, steady-state conditions would correspond to the case where $\frac{dv(t)}{dt} = \frac{mg - cv(t)}{m} = 0$. Figure A1.2 is plotted by using the solution, $v(t) = \frac{gm}{c}\left(1 - e^{-(c/m)t}\right)$. From the Figure 1.2 below, it shows that the falling parachutist's velocity keeps increasing and thus varies with time initially (transient case). When the velocity reaches to a point where the $F_{gravity}, (mg)$ is equal to $F_{drag}, (cv)$, there is no more increase in velocity afterward due to $\frac{dv(t)}{dt} = 0$.

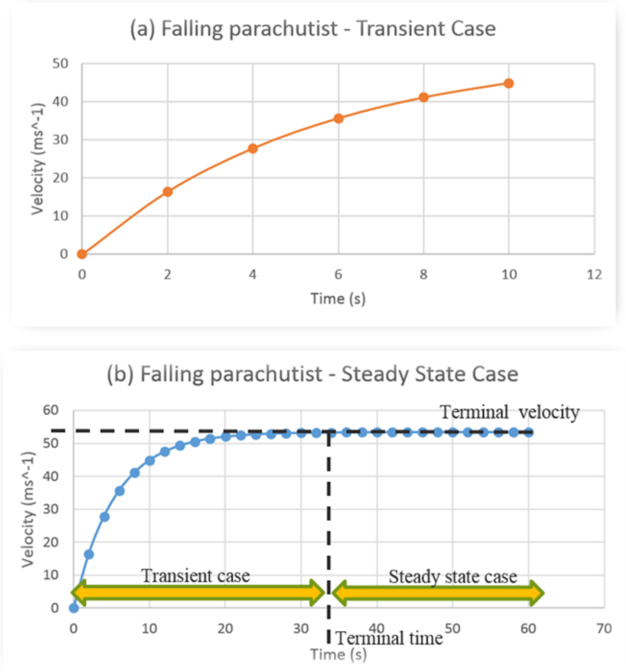


Figure A1.2: (a) Transient and (b) steady state cases for falling parachutist problem.

From Figure A1.2(b), the terminal velocity and terminal time can be computed as $v_{terminal} = \frac{gm}{c}$ and $t_{terminal} = -\frac{m}{c}\ln\left|1 - \frac{c}{gm}(v_{terminal})\right|$. Note that velocity will increase when the first derivative, $\frac{dv(t)}{dt} > 0$, decrease when $\frac{dv(t)}{dt} < 0$, and remain unchanged when $\frac{dv(t)}{dt} = 0$.

APPENDIX 1.3 HOMOGENEITY OF 1ST ORDER NON-LINEAR ODE

(i) *Homogeneous and non-homogeneous equations of* $\left(\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}\right)$

This section is particular important especially when we deal with 1st order nonlinear ODE problem. It is worthwhile to mention that there is another method to classify the homogeneous and non-homogeneous groups in ODE. For 1st order ODE equation, the classification is given below.

First of all, the descriptions of the homogeneous and non-homogeneous functions are given:

A *homogeneous function* $f(x, y)$ is said to be *homogeneous of degree* n, if we get

$$f(\lambda x, \lambda y) = \lambda^n f(x, y)$$

for all arbitrary constant λ.

Non-homogeneous function is

any function that does not follow the homogeneous format as equation above, i.e.

$$f(\lambda x, \lambda y) \neq \lambda^n f(x, y)$$

Example (1): Check the homogeneous degree for the function $x^4 + xy^3$.

In this case, $f(x, y) = x^4 + xy^3$.

Applying $f(\lambda x, \lambda y)$, we get $f(\lambda x, \lambda y) = (\lambda x)^4 + (\lambda x)(\lambda y)^3 = \lambda^4(\lambda + xy^3)$

Since $f(\lambda x, \lambda y) = \lambda^n f(x, y)$, thus we say the function $x^4 + xy^3$ is *homogeneous of degree*, $n = 4$ (i.e. homogeneous function with degree 4)

Example (2): Check the homogeneous degree for the function $y^2 - xy + 1$.

In this case, $f(x, y) = y^2 - xy + 1$.

Applying $f(\lambda x, \lambda y)$, we get $f(\lambda x, \lambda y) = (\lambda y)^2 - (\lambda x)(\lambda y) + 1$.

Since $f(\lambda x, \lambda y) \neq \lambda^n f(x, y)$ or $(\lambda y)^2 - (\lambda x)(\lambda y) + 1 \neq \lambda^2 (y^2 - xy + 1)$, thus we say the function $(y)^2 - (xy) + 1$ is *nonhomogeneous*.

Then, the homogeneous and nonhomogeneous differential equation of 1st order ODE are given:

Case (1): Homogeneous differential equation of $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$
$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ is a homogeneous differential equation if $f(x,y)$ and $g(x,y)$ are *homogeneous of the same degree*.
For example: $\frac{dy}{dx} = \frac{y^2-x^2}{2xy}$ It is a *1st order nonlinear homogeneous differential equation* where the functions at numerator (i.e. $y^2 - x^2$) and denominator (i.e. $2xy$) are homogeneous of degree 2.

Case (2): Nonhomogeneous differential equation of $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$
$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ is a nonhomogeneous differential equation if $f(x,y)$ and $g(x,y)$ are nonhomogeneous or they have homogeneity with different degrees.
For example [1]: $\frac{dy}{dx} = \frac{2x-4y+5}{x-2y+3}$ It is a *1st order nonlinear nonhomogeneous differential equation* where the functions at numerator (i.e. $2x - 4y + 5$) and denominator (i.e. $x - 2y + 3$) are nonhomogeneous. *For example [2]:* $\frac{dy}{dx} = \frac{y^2-x^2}{2x^2y^2}$ It is a *1st order nonlinear nonhomogeneous differential equation* where the functions at numerator and denominator are homogeneous of degree 2 and degree 4 respectively.

To avoid confusion, we use the term "homogeneous/nonhomogeneous" for all the linear ODE case, while we use the term "homogeneous/ nonhomogeneous of $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ form" for the 1^{st} order nonlinear ODE case. The homogeneity of 2^{nd} and higher order nonlinear ODE is out of scope in this study.