APPENDIX 2.1: EXACT DIFFERENTIAL EQUATION

Some differential equations are of a form that can be solved readily, it would be useful to be able to recognize them. For example: we know the derivative for:

Product rule	Quotient rule
$\frac{d(xy)}{dx} = x\frac{d(y)}{dx} + y\frac{d(x)}{dx} = xy' + y$	$\frac{d(\frac{x}{y})}{dx} = \frac{y\frac{d(x)}{dx} - x\frac{d(y)}{dx}}{y^2} = \frac{y - xy'}{y^2}$

Example: Solve the differential equation $xy' + y = 0$.

Solution: If we recalled the product rule, we can see that the LHS is equal to $xy' + y = \frac{d(xy)}{dx}$. Thus, the differential equation $xy' + y = 0$ is of the 'exact' form. Hence, we can further solve it as

$xy' + y = 0$

>> $\frac{d(xy)}{dx} = 0$

>> $\int \frac{d(xy)}{dx} dx = \int 0 dx$

>> $xy = C$

>> $y = \frac{C}{x}$

Identification of exact differential equation would be difficult in most of the time as it involves a lot of memorization and the identification process become hard, especially when the differential equation is varies with the original format that you memorize.

For example: Is $xy' + y + 4 = 0$ an exact differential equation ?

Thus, it is crucial for us to know the equation whether it is exact or not before we proceed with some lengthy process in finding its solution. In general, exact differential equation can be identified by using the following approach.

Procedure to check the exact differential equation:

First, arrange the 1st order differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

where $M(x, y)$ & $N(x, y)$ are two functions of independent variable x and dependent variable y.

If $\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$,

then it is an *exact differential equation*.

If $\frac{\partial M(x,y)}{\partial y} \neq \frac{\partial N(x,y)}{\partial x}$,

then there is no way for the differential equation to be exact.

Thus, we need to use other strategy to solve the problem

Example: Solve the differential equation $xy' + y + 4 = 0$.

Solution: Check if we can solve using exact differential equation.

$xy' + y + 4 = 0$

>> $xdy + (y + 4)dx = 0$ [***Comment***: *Arrange it in the form of* $(x, y)dx + N(x, y)dy = 0$]

>> where

$M(x, y) = (y + 4)$

$N(x, y) = x$

>> Check it with $\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$ condition.

LHS: $\frac{\partial M(x,y)}{\partial y} = \frac{\partial}{\partial y}(y + 4) = 1$

RHS: $\frac{\partial N(x,y)}{\partial x} = \frac{\partial}{\partial x}(x) = 1$

∴ Since $\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$ is true, we can solve it using *exact differential equation*.

Further solve it using exact approach.

$xy' + y + 4 = 0$

>> $\frac{d(x(y+4))}{dx} = 0$

>> $\int \frac{d(x(y+4))}{dx} dx = \int 0 dx$

>> $x(y + 4) = C$

>> $y = \frac{C}{x} - 4$

Previous examples show that if the differential equation can be in the "exact" form, it can be solved directly. However, it is *undesired to obtain the exact differential equation* in most of the time.

Example: Solve the differential equation $xy' - y = 0$

Solution:

Again, we are not sure if $xy' - y = 0$, so we check it with $\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$ condition.

$xy' - y = 0$

>> $xdy - ydx = 0$ [***Comment:*** *Arrange it in the form of* $(x, y)dx + N(x, y)dy = 0$]

>> where

$M(x, y) = -y$

$N(x, y) = x$

>> Check it with $\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$ condition.

LHS: $\frac{\partial M(x,y)}{\partial y} = \frac{\partial}{\partial y}(-y) = -1$

RHS: $\frac{\partial N(x,y)}{\partial x} = \frac{\partial}{\partial x}(x) = 1$

∴ Since $\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$ is not true, we cannot solve it using exact differential equation.

Note: It can't be solved as it is in "non exact form". However, there is a method used to convert it to exact form. It is a strategy involving **Integrating Factor** and **Linear Differential Equation as shown in** *Section 2.2..1*.

APPENDIX 2.2 BERNOULLI'S DIFFERENTIAL EQUATION

There are some cases cannot be solved by the previous strategies due to its non-linear and non-separable characteristics, however, if it is in the form of Bernoulli's equation, it can be ***converted to the linear differential equation*** and hence solved by using the Integrating Factor strategy.

A *Bernoulli's differential equation* has the form of

$$\frac{dy}{dx} + p(x)y = r(x)y^n$$

Note: The component y^n decides whether it is linear or nonlinear differential equation.

For $n = 0$, $\frac{dy}{dx} + p(x)y = r(x)$ [***linear** differential equation*]

For $n = 1$, $\frac{dy}{dx} + \big(p(x) - r(x)\big)y = 0$ [***linear** differential equation*]

For $n > 1$, $\frac{dy}{dx} + p(x)y = r(x)y^n$ [***nonlinear** differential equation*]

To convert the nonlinear Bernoulli's equation into linear form, we need two important properties:

[i] *Let* $v(x) = \{y(x)\}^{1-n}$

where $y = dependent\ variable$ & $x = independent\ variable$

[ii] The derivative, $\frac{1}{(1-n)}\frac{dv(x)}{dx} = \{y(x)\}^{-n}\frac{dy(x)}{dx}$

Prove:

$\frac{dv(x)}{dx} = (1-n)\{y(x)\}^{(1-n)-1}\frac{dy(x)}{dx}$

>> $\frac{1}{(1-n)}\frac{dv(x)}{dx} = \{y(x)\}^{-n}\frac{dy(x)}{dx}$

Substitute eqns. [i] & [ii] to the nonlinear Bernoulli's equation:

For $n > 1$, $\frac{dy}{dx} + p(x)y = r(x)y^n$ [***Comment**: nonlinear Bernoulli's equation*]

where $y = dependent\ variable$ & $x = independent\ variable$

Divide by y^n and we get $y^{-n}\frac{dy}{dx} + p(x)y^{1-n} = r(x)$ [***Step 1**: Divide by* y^n]

Substitute the eqns.

(i) $v(x) = \{y(x)\}^{1-n}$;

(ii) $\frac{1}{(1-n)}\frac{dv(x)}{dx} = \{y(x)\}^{-n}\frac{dy(x)}{dx}$.

We get [***Step 2**: Substitution method*]

>> $\frac{1}{(1-n)}\frac{dv(x)}{dx} + p(x)v(x) = r(x)$ [***Comment**: linear Bernoulli's equation*]

where $v = dependent\ variable$ & $x = independent\ variable$

Hence, the 1st order linear differential equation can be solved using the previous strategy. Once you obtain the solution v. Back substitute it into eqn. $v(x) = \{y(x)\}^{1-n}$ to obtain the solution y.

For example: Solve the equation $\frac{dy}{dx} + 3x^2y = xe^{x^3}y^2$

Solution:

$\frac{dy}{dx} + 3x^2y = xe^{x^3}y^2$

Think: What is the nonlinear component.

Recognize that it is nonlinear Bernoulli eqn. $\frac{dy}{dx} + p(x)y = r(x)y^n$,

With $p(x) = 3x^2$, $r(x) = xe^{x^3}$, $n = 2$

[***Step 1**: Divide by y^n*]

>> $y^{-2}\frac{dy}{dx} + 3x^2y^{-1} = xe^{x^3}$

[***Step 2**: Substitution method*]

>> Let $v(x) = \{y(x)\}^{1-2} = \{y(x)\}^{-1}$;

>> Let $\frac{1}{(1-n)}\frac{dv(x)}{dx} = \{y(x)\}^{-2}\frac{dy(x)}{dx} = -\frac{dv(x)}{dx}$.

$y^{-2}\frac{dy}{dx} + 3x^2y^{-1} = xe^{x^3}$

>> $-\frac{dv(x)}{dx} + 3x^2v(x) = xe^{x^3}$ [***Comment**: linear Bernoulli's equation*]

[***Step 3**: Linear Differential equation*]

$-\frac{dv(x)}{dx} + 3x^2v(x) = xe^{x^3}$

>> $\frac{dv(x)}{dx} - 3x^2v(x) = -xe^{x^3}$ [***Step 1***- *Linear Form*]

where

$v = dependent\ variable$

$x = independent\ variable$

$p(x)$=$-3x^2$

$r(x) = -xe^{x^3}$

>> The integrating factor, $IF = e^{\int p(x)dx} = e^{\int -3x^2dx} = e^{-x^3}$ [***Step 2***- *IF*]

>> $e^{-x^3}\frac{dv(x)}{dx} - e^{-x^3}(3x^2v(x)) = e^{-x^3}(-xe^{x^3})$ [***Step 3***- *Multiply*]

>> $e^{-x^3}\frac{dv(x)}{dx} - e^{-x^3}\left(3x^2v(x)\right) = -x$

>> $\frac{d}{dx}\left(e^{-x^3}.v(x)\right) = -x$ [***Step 4***- *Exact*]

where $IF = e^{-x^3}$ & $v = dependent\ variable$

>> $\int \frac{d}{dx}\left(e^{-x^3}.v(x)\right)dx = \int(-x)dx$ [***Step 5***- *Integrate*]

>> $e^{-x^3}.v(x) = \frac{-x^2}{2} + C$

[***Step 4***: *Back Substitution*]

>> Previously we obtained $v(x) = \{y(x)\}^{-1}$ and $e^{-x^3}.v(x) = \frac{-x^2}{2} + C$

>> $e^{-x^3}.\{y(x)\}^{-1} = \frac{-x^2}{2} + C$

$\therefore y(x) = e^{-x^3}.\frac{1}{\frac{-x^2}{2}+C} = \frac{2}{e^{x^3}(2C-x^2)}$, where C = arbitrary constant.

For example: Solve the equation $3\frac{dy}{dx} + y = (1-2x)y^4$.

Solution:

$3\frac{dy}{dx} + y = (1-2x)y^4$

Think: What is the nonlinear component.

Recognize that it is nonlinear Bernoulli eqn. $\frac{dy}{dx} + p(x)y = r(x)y^n$,

With $p(x) = \frac{1}{3}$, $r(x) = \frac{(1-2x)}{3}$, $n = 4$

[***Step 1**: Divide by y^n*]

$\frac{dy}{dx}+\frac{y}{3}=\frac{(1-2x)}{3}y^4$

>> $y^{-4}\frac{dy}{dx}+\frac{y^{-3}}{3}=\frac{(1-2x)}{3}$

[***Step 2**: Substitution method*]

>> Let $v(x)=\{y(x)\}^{1-4}=\{y(x)\}^{-3}$;

>> Let $\frac{1}{(1-n)}\frac{dv(x)}{dx}=\{y(x)\}^{-4}\frac{dy(x)}{dx}=\frac{1}{-3}\frac{dv(x)}{dx}$.

$y^{-4}\frac{dy}{dx}+\frac{y^{-3}}{3}=\frac{(1-2x)}{3}$

>> $\frac{1}{-3}\frac{dv(x)}{dx}+\frac{v(x)}{3}=\frac{(1-2x)}{3}$

>> $-\frac{dv(x)}{dx}+v(x)=(1-2x)$ [***Comment**: linear Bernoulli's equation*]

[***Step 3**: Linear Differential equation*]

$-\frac{dv(x)}{dx}+v(x)=(1-2x)$

>> $\frac{dv(x)}{dx}-v(x)=(2x-1)$ [***Step a**- Linear Form*]

where

$v=dependent\ variable$

$x=independent\ variable$

$p(x)=-1$

$r(x)=(2x-1)$

>> The integrating factor, $IF=e^{\int p(x)dx}=e^{\int -1dx}=e^{-x}$ [***Step b**- IF*]

>> $e^{-x}\frac{dv(x)}{dx}-e^{-x}v(x)=e^{-x}(2x-1)$ [***Step c**- Multiply*]

>> $\frac{d}{dx}\left(e^{-x}.v(x)\right)=e^{-x}(2x-1)$ [***Step d**- Exact*]

where $IF=e^{-x}$ & $v=dependent\ variable$

>> $\int\frac{d}{dx}\left(e^{-x}.v(x)\right)dx=\int e^{-x}(2x-1)dx$ [***Step e**- Integrate*]

>> $e^{-x}.v(x)=\int e^{-x}(2x)-e^{-x}dx$

Where $\int e^{-x}(2x)-e^{-x}dx=\int e^{-x}(2x)-2e^{-x}+e^{-x}dx$

$$= \int e^{-x}(2x) - 2e^{-x}dx + \int e^{-x}dx$$

$$= -2xe^{-x} + (-e^{-x}) + C$$

Hint: Product rule, i.e. $\int f'(x)g(x) + f(x)g'(x)\, dx = f(x)g(x) + C$

Think: Can we use integration by part, i.e. $\int u\, dv = uv - \int v\, du$ to solve the integration problem?

>> $e^{-x}. v(x) = -2xe^{-x} + (-e^{-x}) + C$

[***Step 4**: Back Substitution*]

>> Previously we obtained $v(x) = \{y(x)\}^{-3}$ and $e^{-x}. v(x) = -2xe^{-x} - e^{-x} + C$

>> $e^{-x}. \{y(x)\}^{-3} = -2xe^{-x} - e^{-x} + C$

>> $\{y(x)\}^3 = e^{-x}. \frac{1}{-2xe^{-x}-e^{-x}+C} = \frac{1}{-2x-1+Ce^{x}}$

$\therefore y(x) = \sqrt[3]{\frac{1}{-2x-1+Ce^{x}}}$, where C = arbitrary constant.

APPENDIX 2.3DIFFERENTIAL EQUATION OF HOMOGENEOUS $dy/dx = f(x,y)/g(x,y)$ FORM

Previously, we use the $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ form to check the homogeneity of 1st order nonlinear ODE. This section introduces the strategy used to solve the homogeneous $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$.

A homogeneous differential equation has the form of $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ if $f(x, y)$ and $g(x, y)$ are homogeneous of the same degree, where $f(\lambda x, \lambda y) = \lambda^{n_1} f(x, y)$ $g(\lambda x, \lambda y) = \lambda^{n_2} g(x, y)$ $\& \; n_1 = n_2$

<u>For example</u>: Solve $2xy\frac{dy}{dx} - y^2 = -x^2$

Think: What is the nonlinear component? Is it in Bernoulli eqn. form?

Hint: If it is non-linear and can't be in Bernoulli eqn. form, we need to use other method to solve the 1st order nonlinear ODE equation.

Rearrange it to the form of $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$, we get

>> $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$

Check its homogeneous degree

>> $\frac{f(\lambda x,\lambda y)}{g(\lambda x,\lambda y)} = \frac{(\lambda y)^2 - (\lambda x)^2}{2(\lambda x)(\lambda y)} = \frac{\lambda^2((y)^2 - (x)^2)}{\lambda^2(2(xy))} = \frac{\lambda^2 f(x,y)}{\lambda^2 g(x,y)}$

∴ It is a 1st order nonlinear ODE with homogeneous $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ form of degree 2.

The 1st order nonlinear ODE with homogeneous $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ form can be solved by using substitution method as shown in the table below. In general, this form is very useful to ***convert the non-separable differential equation into separable differential equation***.

To convert the 1st order nonlinear ODE with homogeneous $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ form into separable form, we need two important properties:

[i] *Let* $v(x) = \frac{y}{x}$ *or* $y = vx$

where $y = dependent\ variable$ & $x = independent\ variable$

[ii] The derivative, $\frac{dy}{dx} = x\frac{d}{dx}[v] + v$

For example: Solve $2xy\frac{dy}{dx} - y^2 = -x^2$

Solution:

$2xy\frac{dy}{dx} - y^2 = -x^2$

>> $\frac{dy}{dx} = \frac{y^2-x^2}{2xy}$

[***Comment****:* 1st order nonlinear ODE with homogeneous $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ form*; non-separable form*]

Step 1*: Substitution method*

$y = vx$

$\frac{dy}{dx} = x\frac{d}{dx}[v] + v$

>> $\frac{dy}{dx} = \frac{(vx)^2-x^2}{2x(vx)} = x\frac{d}{dx}[v] + v$

>> $\frac{(v)^2-1}{2(v)} - v = x\frac{dv}{dx}$ [***Comment****: Solve using separable differential equation*]

>> $\frac{1}{x}dx = \frac{1}{\frac{(v)^2-1}{2(v)}-v}dv$

>> $\frac{1}{x}dx = \frac{1}{\frac{v^2-1-2v^2}{2(v)}}dv$

>> $\frac{1}{x}dx = \frac{2v}{-v^2-1}dv$ [***Step a****- Separable form*]

>> $\int\frac{1}{x}dx = \int\frac{2v}{-v^2-1}dv$ [***Step b****- Integrate both sides*]

>> $ln|x| = -ln|-v^2-1| + C$

Hint: Solve $\int\frac{2v}{-v^2-1}dv$ using substitution method, let $u = -v^2 - 1$

Step 2*: Back Substitution*

>> Previously we obtained $y = vx$ & $ln|x| = -ln|-v^2-1| + C$

>> $ln|x| = -ln\left|-\left(\frac{y}{x}\right)^2 - 1\right| + C$

>> $ln|x| + ln\left|-\left(\frac{y}{x}\right)^2 - 1\right| = +C$

>> $ln\left|-\frac{y^2}{x} - x\right| = +C$

>> $-\frac{y^2}{x} - x = e^C$

>> $y^2 = -x^2 - xe^C$

$\therefore\ y = \pm\sqrt{-x^2 - xe^C}$

APPENDIX 2.4 DIFFERENTIAL EQUATION OF NONHOMOGENEOUS $dy/dx = f(x,y)/g(x,y)$ FORM

Previous strategies/ methods are not sufficient to solve the nonhomogeneous $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ problem such as $\frac{dy}{dx} = \frac{2x-4y+5}{x-2y+3}$. It involves other types of substitution to solve the problem. For example, for the nonhomogeneous $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$, we use the substitution as follows:

Case I $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$	Case II $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$
Let $v = a_1x + b_1y$	Let $\begin{matrix} x = X + h \\ y = Y + k \end{matrix}$ where the pair constant (h, k) can be obtained by solving the simultaneous equations: $a_1h + b_1k + c_1 = 0$ $a_2h + b_2k + c_2 = 0$
The substitution converts the nonhomogeneous $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ to homogeneous form and hence it can be solved by method introduced in Appendix 11.7.	

The nonlinear nonhomogeneous $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ has a lot of branches and it may require various types of substitution to convert it to homogeneous form. The topic is wide and thus it is **not covered** in this study.

For example: Solve $\frac{dy}{dx} = \frac{2x-4y+5}{x-2y+3}$

Solution:

$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} = -4 - (-4) = 0$ [***Comment***: *Case I*]

$v = 2x - 4y$ → **RHS:** $\frac{2x-4y+5}{x-2y+3} = \frac{v+5}{\frac{v}{2}+3}$

$\frac{dv}{dx} = 2 - 4\frac{dy}{dx}$ → **LHS:** $\frac{dy}{dx} = \frac{2-\frac{dv}{dx}}{4}$

RHS= LHS → $\frac{v+5}{\frac{v}{2}+3} = \frac{2-\frac{dv}{dx}}{4}$ → $\frac{dv}{dx} = 2 - \frac{8v+40}{v+6} = \frac{6v+28}{v+6}$ [***Comment***: *Separable form*]

$\frac{dv}{dx} = 2 - \frac{8v+40}{v+6} = \frac{6v+28}{v+6}$

$\int \frac{v+6}{6v+28} dv = \int dx$

→ $\int \frac{1}{6} + \frac{4}{3(6v+28)} dv = \int dx$

→ $\frac{1}{6}v + \frac{4}{3}\frac{ln|6v+28|}{6} = x + C$

→ $\frac{1}{6}(2x - 4y) + \frac{2}{9}ln|6(2x - 4y) + 28| = x + C$ [***Comment***: *Back substitution of* $v = 2x - 4y$]

∴ $(\frac{1}{3}x - \frac{2}{3}y) + \frac{2}{9}ln|12x - 24y + 28| = x + C$ [***General implicit solution***]

For example: Solve $\frac{dy}{dx} = \frac{-x+2y-4}{2x-y+2}$

Solution:

$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} = 1 - (-4) = -3 \neq 0$ [***Comment***: *Case II*]

$-h + 2k - 4 = 0$

$2h - k + 2 = 0$ → **Solving 2 eqns and obtain** $h = 0, k = 2$

$x = X$ & $y = Y + 2$ → **LHS:** $\frac{-x+2y-4}{2x-y+2} = \frac{-X+2(Y+2)-4}{2X-(Y+2)+2} = \frac{-X+2Y}{2X-Y}$

$dx = dX$ & $dy = dY$ → **RHS:** $\frac{dy}{dx} = \frac{dY}{dX}$

RHS= LHS → $\frac{dY}{dX} = \frac{-X+2Y}{2X-Y}$ [***Comment***: *Homogeneous form*]

$Y = vX$ → $\frac{dY}{dX} = \frac{-X+2vX}{2X-vX}$

$\frac{dY}{dX} = v + X\frac{dv}{dX}$ → $v + X\frac{dv}{dX} = \frac{-X+2vX}{2X-vX}$

→ $X\frac{dv}{dX} = \frac{-X+2vX}{2X-vX} - v = \frac{-1+v^2}{2-v}$ [***Comment***: *Separable form*]

$\int \frac{v+6}{6v+28} dv = \int dx$

→ $\int \frac{2-v}{-1+v^2} dv = \int \frac{dX}{X}$

→ $\int \frac{-3}{2(v+1)} + \frac{1}{2(v-1)} dv = \int \frac{dX}{X}$ [***Partial fraction***]

→ $-\frac{3}{2}ln|v + 1| + \frac{1}{2}ln|v - 1| = ln|X| + C$

→ $-\frac{3}{2}ln\left|\frac{Y}{X} + 1\right| + \frac{1}{2}ln\left|\frac{Y}{X} - 1\right| = ln|X| + C$ [***Comment***: *Back substitution of* $Y = vX$]

→ $-\frac{3}{2}ln\left|\frac{y-2}{x} + 1\right| + \frac{1}{2}ln\left|\frac{y-2}{x} - 1\right| = ln|x| + C$ [***Comment***: *Back substitution of* $x = X$, $y = Y + 2$]

∴ $-\frac{3}{2}ln\left|\frac{y-2}{x} + 1\right| + \frac{1}{2}ln\left|\frac{y-2}{x} - 1\right| = ln|x| + C$ [***General implicit solution***]

APPENDIX 2.5 OTHER TYPES OF INTEGRATING FACTOR

Condition	Integrating factor (IF)
If $M(x,y)dx + N(x,y)dy = 0 is$ a homogeneous equation in x & y.	$IF = \frac{1}{Mx+Ny}, \quad Mx + Ny \neq 0$
If $M(x,y)dx + N(x,y)dy = 0$ is of the form $f(xy)ydx + \Phi(xy)dy + N(x,y)dy = 0$	$IF = \frac{1}{Mx-Ny}, \quad Mx - Ny \neq 0$
If $M(x,y)dx + N(x,y)dy = 0$ be a differential equation. If $\frac{\partial M(x,y)/\partial y - \partial N(x,y)/\partial x}{N}$ is a function of x only, i.e. $f(x)$	$IF = e^{\int f(x)dx}$
If $M(x,y)dx + N(x,y)dy = 0$ be a differential equation. If $\frac{\partial N(x,y)/\partial x - \partial M(x,y)/\partial y}{M}$ is a function of y only, i.e. $f(y)$	$IF = e^{\int f(y)dx}$
For the equation $x^a x^b(mydx + nxdy) + x^{a'} x^{b'}(m'ydx + n'xdy) = 0$	$IF = x^h y^k$, where h & k are such that $\frac{a+h+1}{m} = \frac{b+k+1}{n}$; $\frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$.

Note: This is for your extra knowledge and other types of integrating factors are not included in this study.