FIRST ORDER DIFFERENTIAL EQUATIONS

WEEK 1: FIRST ORDER DIFFERENTIAL EQUATIONS

1.1 Introduction to differential equation

If we want to solve an engineering problem (usually of a physical nature), we first have to formulate the problem as a mathematical expression in terms of variables, functions, and equations. Such an expression is known as a mathematical model of the given problem. The process of setting up a model, solving it mathematically, and interpreting the result in physical or other terms is called mathematical modeling or briefly modeling.

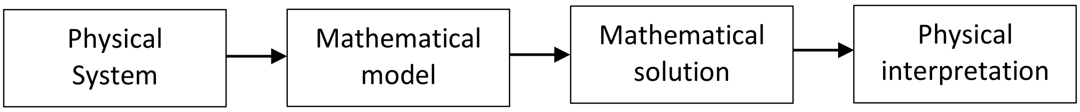
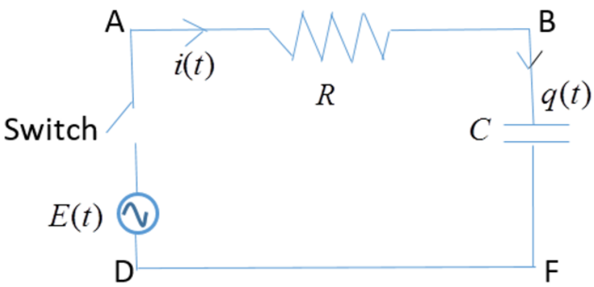


Figure 1: Modeling, solving, interpreting

In creating a mathematical model of a physical system, we frequently involve differential equation/ integral equation / integro-differential equations to express relationships, such as ‘the force acting on a falling object is proportional to its acceleration’, ‘voltage drop across a resistor is proportional to the current’, etc.

Table 1.1: Three types of equations of a mathematical model: (i) Differential equation (ii) Integral equation (iii) Integro-differential equation.

<table>
<tr><th>(i) Differential equation</th><th>(ii) Integral equation</th><th>(iii) Integro-differential equation</th></tr>
<tr><td>Equations which involve <u>derivatives</u> of the variables in the model.</td><td>Equations which involve <u>integrals</u> of the variables in the model.</td><td>Equations which involve <u>both</u> <u>derivatives and integrals</u> of the variables in the model.</td></tr>
<tr><td colspan="3"><u>For example: RC Electrical Circuit</u>

What are the amounts of charge and current flow in an electric circuit that consists a generator (E volt), a resistance (R ohms) and a capacitor (C capacitance)? The RC electrical circuit is shown below.

</td></tr>
</table>

From experiments, we know that the voltage loss through a resistor and capacitor is proportional to the current and charge respectively, where $\Delta V_{resistor}(t) \propto i(t)$ and $\Delta V_{capacitor}(t) \propto q(t)$. Hence, $\Delta V_R(t) = Ri(t)$ and $\Delta V_C(t) = \frac{1}{C}q(t)$.

According to Kirchhoff's voltage law, the summation of voltage in a closed loop is equal to zero. Thus, $(V_B - V_A) + (V_F - V_B) + (V_D - V_F) + (V_A - V_D) = 0$

$$Ri(t) + \frac{1}{C}q(t) + 0 + (-E(t)) = 0$$

$$Ri(t) + \frac{1}{C}q(t) = E(t)$$

From definition, the current is equal to the rate of charge flow or the charge is the integral of the current over time.

$i(t) = \frac{dq(t)}{dt}$ or $q(t) = \int i(t)dt$

Rearrange it, we obtain three different forms of mathematical model as shown below, to solve the desired variables (i.e. charge and current). Note that different methods and strategies are used to solve these equations. In this study, we will focus on the topic of differential equation.

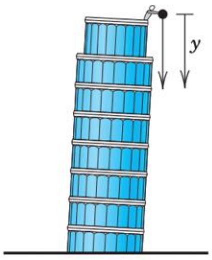
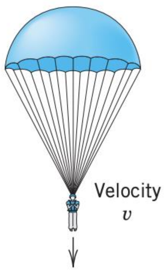
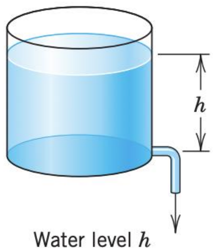
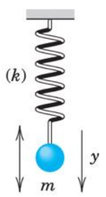
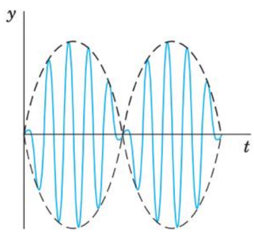
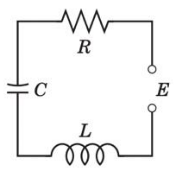
(i) $R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = E(t)$	*(ii)* $Ri(t) + \frac{1}{C}\int i(t)dt = E(t)$	*(iii)* $R\frac{dq(t)}{dt} + \frac{1}{C}\int i(t)dt = E(t)$
-involves derivative, $\frac{dq(t)}{dt}$. -This is known as differential equation.	-involves integrals, $\int i(t)dt$. -This is known as integral equation.	-involves both derivative, $\frac{dq(t)}{dt}$ and integrals, $\int i(t)dt$. -This is known as integro-differential equation.

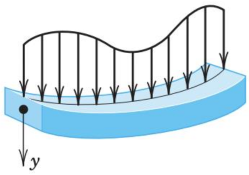
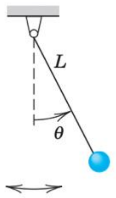
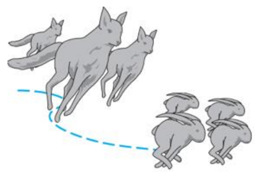
Differential equation (DE) plays a fundamental role in engineering because many physical phenomena are best formulated mathematically in terms of their rate of change. By solving a differential equation, we can gain a deeper understanding of the physical processes that these equations are describing. Some examples of fundamental laws that are written in terms of the rate of change of variables are shown in the table below.

Table 1.2: Examples of fundamental laws written in the differential equation.

Fundamental Law	**Mathematical Expression**	**Variables and Parameter**
Newton's 2nd Law of Motion	$\frac{dv(t)}{dt} = \frac{\Sigma F(t)}{m}$	Velocity (v); Force (F); Mass (m).
Faraday's Law (Voltage drop across an inductor)	$\Delta v_L(t) = L\frac{di(t)}{dt}$	Voltage drop ($\Delta v_L(t)$); Inductance (L) Current (i).
Fourier's Heat Law	$q(x) = -k'\frac{dT(x)}{dx}$	Heat flux (q); Thermal conductivity (k'); Temperature (T).
Fick's law of diffusion	$J(x) = -D\frac{dC(x)}{dx}$	Mass flux (J); Diffusion coefficient (D); Concentration(C).

Table 1.3: Examples of some application of differential equations

 Falling stone $y'' = g = const.$	 Parachutist $mv' = mg - bv^2$	 Water level h Outflowing water $h' = -k\sqrt{h}$
 Displacement y Vibrating mass on a spring $my'' + ky = 0$	 Beats of a vibrating system $y'' + \omega_0^2 y = \cos \omega t, \quad \omega_0 \approx \omega$	 Current I in an RLC circuit $LI'' + RI' + \frac{1}{C}I = E'$

 Deformation of a beam $EIy^{iv} = f(x)$	 Pendulum $L\theta'' + g\sin\theta = 0$	 Lotka–Volterra predator–prey model $y_1' = ay_1 - by_1y_2$ $y_2' = ky_1y_2 - ly_2$

Hint: The use of differential equations may empower us *to make precise predictions about the future behaviour of our models/ system*. The motivation and implementation of the mathematical modelling with differential equation in the engineering problem solving is illustrated in Appendices 1.1 & 1.2.

1.2 The classification/type of differential equations

Different types of differential equations may require different strategies to solve the problem. Thus, it is important for the user to understand, recognize and classify the correct categories of differential equations.

(i) *Independent and dependent variables*

A differential equation expresses such that the dependent variable(s) depends on the independent variable.

Dependent variable	**Independent variable**
It is the variable(s) that is differentiated.	It is the variable(s) with respect to which differentiation occurs.
Example (1): $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} = cos2x$ This differential equation has *dependent variable* of y and *independent variable* of x. **Note:** The variable y is in the function of x, i.e. $y = y(x)$. In other words, y is changed with respect to x.	
Example (2):	

$\frac{d^2x}{dt^2} - 4x\frac{dx}{dt} = cos2t$ This differential equation has *dependent variable* of x and *independent variable* of t. **Note:** The variable x is in the function of t, i.e. $x = x(t)$. In other words, x is changed with respect to t.

(ii) *Ordinary Differential Equation (ODE) versus Partial Differential Equation (PDE)*

Differential equation can be categorized into 2 cases: ODE & PDE. The classification of ODE and PDE depends on the *number of independent variable*, regardless of the number of dependent variables.

<table>
<tr><th colspan="2">CASE 1: ODE</th></tr>
<tr><td colspan="2">Those equations that involve ordinary derivatives (i.e. d symbol) are called ODE. ODE has only one independent variable. It can be separated into the ODE problem or system of ODE problem depends on the number of dependent variable.</td></tr>
<tr><td>(i) One dependent variable</td><td>(ii) More than one dependent variable</td></tr>
<tr><td>For example: Brine mixture problem

$\frac{dx}{dt} = 2 - \frac{x}{5}$,

where x = concentration of salt.

-This is an ODE problem:

(i) One independent variable (t)

(ii) One dependent variable (x)</td><td>For example: Population of rabbit & fox

$\frac{dx}{dt} = ax - bxy,$

$\frac{dy}{dt} = -cy + dxy,$

where x = rabbit; y = fox.

- This is a system of ODE problem:

(i) One independent variable (t)

(ii) More than one dependent variable (x & y)</td></tr>
<tr><td colspan="2">Comment: Solving ODE problem is the main focus of this study.</td></tr>
</table>

<table>
<tr><th colspan="2">CASE 2: PDE</th></tr>
<tr><td colspan="2">Those equations that involve partial derivatives (i.e. ∂ symbol) are called PDE. <u>PDE has two independent variables or more</u>. It can be separated into the PDE problem or system of PDE problem depends on the number of dependent variable.</td></tr>
<tr><td>(i) One dependent variable</td><td>(ii) More than one dependent variable</td></tr>
<tr><td>For example: Transient heat equation

$\frac{\partial T(x,t)}{\partial t} - \alpha \frac{\partial^2 T(x,t)}{\partial x^2} = 0$,

where T = temperature; α = thermal diffusivity.

-This is a PDE problem:

(i) More than one independent variables (x & t)

(ii) One dependent variable (T)</td><td>For example: Incompressible Navier-Stokes equation for pipe flow

$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} - v \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial \omega}{\partial x_i} = g_i$,

where u = flow velocity; ω = elevation

- This is a system of PDE problem:

(i) More than one independent variables (t , x_j & x_j)

(ii) More than one dependent variables (u_i & ω)</td></tr>
<tr><td colspan="2">Comment: PDE is out of scope in this study. It will be covered in KIX1002. At current stage, students should know how to classify between ODE and PDE.</td></tr>
</table>

(iii) *Order of a differential equation*

The order of a differential equation is the degree of the highest derivative that occurs in the equation. The approach to find the order of ODE and PDE is the same as illustrated below.

<table>
<tr><th>Case 1: Order of ODE</th></tr>
<tr><td>1st order ODE: The order of highest derivative d is 1

Example (1):

$4\frac{dx}{dt} - 3\frac{dy}{dt} - x + y = \cos(2t)$

Example (2):

$(\frac{dx}{dt})^2 - 3\frac{dy}{dt} - x + y = \cos(2t)$</td></tr>
</table>

2^{nd} order ODE: The *order* of highest derivative d^2 is 2 *Example (1):* $\frac{d^2f}{dx^2} - 4x\frac{df}{dx} = \cos(2x)$ *Example (2):* $\frac{d^2f}{dx^2} - 4x\left(\frac{df}{dx}\right)^4 = \cos(2x)$

Case 2: Order of PDE
1^{st} order PDE: The *order* of highest derivative ∂ is 1 *Example (1):* $4\frac{\partial f}{\partial t} - 3\frac{\partial f}{\partial t} - x + y = \cos(2t)$ *Example (2):* $\frac{\partial x}{\partial t} + (\frac{\partial x}{\partial y})^4 = \cos(2t) + 2y$
2^{nd} order PDE: The *order* of highest derivative ∂^2 is 2 *Example (1):* $4\frac{\partial f}{\partial x} - 3\frac{\partial^2 f}{\partial y^2} - x + y = \cos(2t)$ *Example (2):* $\frac{\partial^2 f}{\partial t\,\partial y} + (\frac{\partial f}{\partial y})^3 = \cos(2t) + 2y$

Note: The order of an equation is not affected by any power to which the derivatives may be raised.

Moreover, **degree** of a differential equation is the power of the highest order derivative.

Example 1.1:

$$(\frac{dx}{dt})^2 - 3\frac{dy}{dt} - x + y = \cos(2t)$$

The differential equation above is a 1^{st} order ODE with degree 2.

Prove: (i) 1^{st} order because the *order* of highest derivative d is 1

(ii) ODE because it has only one independent variable (t)

(iii) *Degree* 2 because the power of the highest order derivative $(\frac{dx}{dt})^2$ is 2

Example 1.2:

$$\frac{\partial^2 f}{\partial t\, \partial y} + \left(\frac{\partial f}{\partial y}\right)^3 = \cos(2t) + 2y$$

The differential equation above is a 2nd order PDE with degree 1.

Prove: (i) 2nd order because the *order* of highest derivative ∂^2 is 2

(ii) PDE because it has more than one independent variables (t & y)

(iii) *Degree* 1 because the power of the highest order derivative $\frac{\partial^2 f}{\partial t\, \partial y}$ is 1

Example 1.3:

$$\frac{\partial x}{\partial t} + \left(\frac{\partial x}{\partial y}\right)^4 = \cos(2t) + 2y$$

The differential equation above is a 1st order PDE with degree 4.

Prove: (i) 1st order because the *order* of highest derivative ∂ is 1

(ii) PDE because it has more than one independent variables (t & y)

(iii) *Degree* 4 because the power of the highest order derivative $\left(\frac{\partial x}{\partial y}\right)^4$ is 4

(iv) *Linear and nonlinear differential equations*

We used to plot a linear graph of y_1 versus y_2 using $y_1 = my_2 + c$ (Linear Algebraic Eqn), where m & c are the slope and the intercept respectively. In this case, y_1 is in the function of y_2. Therefore, y_1 is the dependent variable while y_2 is the independent variable. Rearrange it, we obtain the general form of linear equation as follows: $a_1 y_1 + a_0 y_2 = c$ where $a_1 = 1$ and $a_0 = -m$.

By using similar approach, we get 1st order linear ODE where $a_1(x)y' + a_0(x)y = g(x)$. In this case, y' is the first derivative of y. Linear ODE has the properties of $f(y_1 + y_2) = f(y_1) + f(y_2)$.

In general, a **linear ODE** of order n^{th} has the following form:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x),$$

where

$a_i(x)$ is a function of independent variable (x) for $i = 0,1, \ldots, n$.

$g(x)$ is a function of independent variable (x)

y is the dependent variable

Any equation of ODE that does not follow the linear format as equation above is known as **nonlinear ODE**. For example:

$$a_n(x,y)(y^{(n)})^A + a_{n-1}(x,y)(y^{(n-1)})^B + \cdots + a_1(x,y)(y')^C + a_0(x,y)(y)^D = g(x,y).$$

where the power $A, B, C \;\&\; D \neq 1$; $a_i(x,y)$ or $g(x,y)$ are functions of dependent variable (y)

Examples of the linear and nonlinear ODEs are given as follow.

Case 1: Linear ODE

General format of **1st order linear ODE**: $a_1(x)y' + a_0(x)y = g(x)$

For example:

1st order linear ODE: $-4\frac{dy}{dx} - x^2 = 0,$

Rearrange it into the general format: $-4\frac{dy}{dx} = x^2$

where

$y' = \frac{dy}{dx}$ $\quad$ ($y = dependent\ variable \;\&\; x = independent\ variable$)

$a_1(x) = -4;$

$a_0(x) = 0;$

$g(x) = x^2.$

∴ It is a *1st order linear ODE* since it follows the linear format: $a_1(x)y' + a_0(x)y = g(x)$

General format of **2nd order linear ODE**: $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$

For example:

2nd order linear ODE: $\frac{d^2f}{dx^2} - 4x\frac{df}{dx} - cos2x - 3 = 0,$

Rearrange it into the general format: $\frac{d^2f}{dx^2} - 4x\frac{df}{dx} = cos2x + 3,$

where

$f'' = \frac{d^2f}{dx^2};$ $\quad$ ($f = dependent\ variable \;\&\; x = independent\ variable$)

$f' = \frac{df}{dx};$

$a_2(x) = 1;$

$a_1(x) = -4x;$

$a_0(x) = 0;$

$g(x) = cos2x + 3$

∴ It is a *2^{nd} order linear ODE* since it follows the linear format:

$$a_2(x)f'' + a_1(x)f' + a_0(x)f = g(x)$$

General format of **3^{rd} order linear ODE**: $a_3(x)y''' + a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$

For example:

3^{rd} order linear ODE: $4\frac{d^3x}{dt^3} + 3\frac{dx}{dt} + 2x + \cos(t) = 2,$

Rearrange it into the general format: $4\frac{d^3x}{dt^3} + 3\frac{dx}{dt} + 2x = 2 - \cos(t),$

where

$x''' = \frac{d^3x}{dt^3};$ $(x = dependent\ variable\ \&\ t = independent\ variable)$

$x' = \frac{dx}{dt};$

$a_3(t) = 4;$

$a_2(t) = 0;$

$a_1(t) = 3;$

$a_0(t) = 2;$

$g(t) = 2 - \cos(t)$

∴ It is a *3^{rd} order linear ODE* since it follows the linear format:

$$a_3(t)x''' + a_2(t)x'' + a_1(t)x' + a_0(t)x = g(t)$$

Case 2: Nonlinear ODE
General format of **1st order linear ODE**: $a_1(x)y' + a_0(x)y = g(x)$ *For example:* 1st order nonlinear ODE: $-4\left(\frac{dy}{dx}\right)^2 = x^2$, Rearrange it into the general format: Same Eqn. ∴ It is a *1st order nonlinear ODE* because it does not obey linear equation: $a_1(x)y' + a_0(x)y = g(x)$ as the derivative $y' \neq \left(\frac{dy}{dx}\right)^2$ where the $\frac{dy}{dx}$ is squared.
General format of **2nd order linear ODE**: $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$ *For example:* 2nd order nonlinear ODE: $\frac{d^2f}{dx^2} - 4f\frac{df}{dx} + cos2x = 0$, Rearrange it into the general format: $\frac{d^2f}{dx^2} - 4f\frac{df}{dx} = -cos2x$ ∴ It is a *2nd order nonlinear ODE* because it does not obey linear equation: $a_2(x)f'' + a_1(x)f' + a_0(x)f = g(x)$ as the $a_1(x) \neq -4f$, where $a_1(x)$ should not be in the function of dependent variable f.
General format of **3rd order linear ODE**: $a_3(x)y''' + a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$ *For example:* 3rd order nonlinear ODE: $4\frac{d^3x}{dt^3} + 3\frac{dx}{dt} + 2\sin(x) + \cos(t) = 2$, Rearrange it into the general format: $4\frac{d^3x}{dt^3} + 3\frac{dx}{dt} + 2\sin(x) = 2 - \cos(t)$,

∴ It is a *3[rd] order nonlinear ODE* because it does not obey linear equation: $a_3(t)x''' + a_2(t)x'' + a_1(t)x' + a_0(t)x = g(t)$ as the $x \neq \sin(x)$, where it has nonlinear sinusoidal function of dependent function x.

Hint 1: Most of the time, nonlinear ODE has <u>nonlinear components</u> such as *coefficient* $a_i(x, y)$ in the function of dependent variable y and the y or its *derivative have degree more than one*, i.e. y^2 & $(\frac{dy}{dx})^3$.

Hint 2: For linear differential equations, there are no products of the dependent variable such as *coefficient* $a_i(x)$ in the function of dependent variable x and its derivatives and neither the derivative occur to any power other than the first power, i.e. y^1 & $(\frac{dy}{dx})^1$.

Extra Info 1: For your additional knowledge, many of the nonlinear equations that occur in engineering cannot be solved easily as they stand, but can be solved, for practical engineering purpose, by the process of replacing them with linear equations that are a close approximation – at least in some region of interest.

Extra Info 2: Sometimes, even if we can't completely solve a differential equation (especially when it deals with nonlinear case), we may still be able to determine useful properties about its solution (qualitative information).

(v) *<u>Homogeneous and nonhomogeneous equations of linear equation</u>*

This is applied to the <u>case of linear differential equation only</u>. Arrange the linear equation in standard format, where all terms containing *dependent variable* occur on left-hand side (***LHS***), while terms containing only the *independent variable and constant* occur on the right-hand side (***RHS***). Linear ODE can be categorized into homogenous and non-homogeneous equation by evaluating the ***RHS*** term as follows.

Case 1: Homogeneous equation
RHS term is equal to zero in the standard format
(i) *<u>Linear homogeneous ODE</u>* *<u>Example (1):</u>* $\frac{dx}{dt} + 4x = 0$ *<u>Example (2):</u>*

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + xsin(t) = 0$$

(ii) *Linear homogeneous PDE*

Example (1):

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 0$$

Example (2):

$$\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial y} = 0$$

Case 2: Nonhomogeneous equation

RHS term is equal to nonzero in the standard format

(i) *Linear nonhomogeneous ODE*

Example (1):

$$\frac{dx}{dt} + 4x = 5$$

Example (2):

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + xsin(t) = \cos(t)$$

(ii) *Linear nonhomogeneous PDE*

Example (1):

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 4x^2 + 2y$$

Example (2):

$\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial y} = 5y$

(vi) *Homogeneous and nonhomogeneous equations of* $(\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)})$

Previous section (*v*) is used to classify the homogeneity of linear ODE. For non-linear ODE, other method is used to classify the homogeneity as shown in *Appendix 1.3*.

This classification is important especially for solving the non-linear ODE problem. Since solving the non-linear ODE problem is out of scope in this study, we will not cover it here.

1.3 Solution to differential equation

The difference between the solution of algebraic equation and differential equation is shown in table below:

Case (1): Solution to Algebraic Equation
(i) We expect the *solution to be a number* For example: Solution of the equation $x + 7 = 10$ is $x = 3$ Or, perhaps, (ii) *Solution to be a set of real & complex numbers* For example: Solution of the polynomial equation $x^3 - 5x^2 + 8x = 12$ are $x_1 = 3.7162$; $x_2 = 0.6419 + i1.6784$ Or, perhaps, (iii) *Solution to be a set of vector or matrix* For example: Solution of two simultaneous equations $x - 5y = 3$ and $3x + 9y = 12$

is a vector $\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 3.625 \\ 0.125 \end{Bmatrix}$.

Case (2): Solution to Differential Equation

The solution of differential equation is not the same as the case of algebraic equation.

(i) We expect the solution to be a function

For example:

Solution of the differential equation $\frac{d^2x}{dt^2} + 25x = 0$

is $x(t) = Asin(5t) + Bcos(5t)$

Or, perhaps,

(ii) Solution to be a family of functions

For example:

Solution of multiple differential equations $\frac{dx}{dt} = 6x - 2y$ and $\frac{dy}{dt} = -3y + 5x$

are $x(t) = Ae^{t(7.2419)} - Be^{t(-0.4419)}$; $y(t) = Ae^{t(4.7016)} - Be^{t(-1.7016)}$

The solution of differential equation can be further divided into two types: (a) General Solution (b) Particular Solution.

Case (1): General solution

The most general function that will satisfy the differential equation *contains one or more arbitrary constants*. Normally the number of arbitrary constants equal to the order of the differential equation.

For example:

The *general solution* of the differential equation $\frac{dx}{dt} = -4x$

is $x(t) = Ae^{-4t}$

where any arbitrary constant A can satisfy the equation.

Hint 1: In fact, general solution indicates that there are an infinite number of solutions to the differential equation unless we are given the specific condition of the problem.

Case (2): Particular solution
Giving particular numerical values to the constants in the general solution results in a particular solution of the equation. Normally, particular solution can be obtained by knowing the initial or boundary condition.
For example: Previously, let the general solution of the differential equation $\frac{dx}{dt} = -4x$ is $x(t) = Ae^{-4t}$ where any arbitrary constant A can satisfy the equation. Given *initial condition* where $x(0) = 2.5$, $$x(t) = Ae^{-4t} = Ae^{-4(0)} = A = 2.5$$ Thus, the *particular solution* of the differential equation $\frac{dx}{dt} = -4x$ which has the value 2.5 when $t = 0$ is $x(t) = 2.5e^{-4t}$. Here, only a specific constant $A = 2.5$ can satisfy the equation. **Think:** What is the particular solution of the problem if the initial condition changes to $(0) = 8$?

Hint 2: The actual solution to a differential solution is the *specific solution that not only satisfies the differential equation, but also satisfies the given initial/boundary conditions*.

The particular solution can be obtained from either "***Boundary-value problem***" or "***Initial-value problem***". It depends on the given specific condition about the value of the solution at a particular point, in addition to the differential equation.

Case (1): Boundary-value problem
All conditions are specified at different values of the *independent variable*, usually at extreme points or boundaries of a system.
For example: The *particular solution* of the differential equation $\frac{d^2x}{dL^2} + 25x = 0$

which has the *boundary conditions* $x(0) = 4$ & $x(10) = 7$ [***Comment:*** x are given at $L = 0$ & 10]

is $(L) = 5.78\sin(5L) + 4\cos(5L)$.

Case (2): Initial-value problem

All conditions are specified at the same value of the *independent variable*.

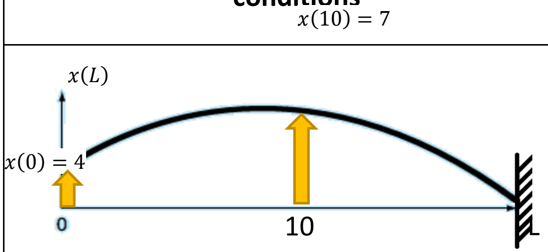
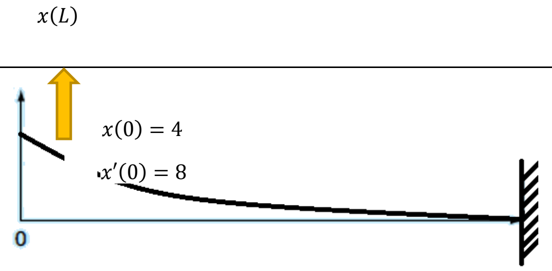
For example:

The *particular solution* of the differential equation $\frac{d^2x}{dt^2} + 25x = 0$

which has the *initial condition* $x(0) = 4$ & $x'(0) = 8$ [***Comment:*** x & x' are given at $t = 0$]

is $x(t) = 1.6\,sin(5t) + 4cos(5t)$.

Graphical representations *of the boundary condition and initial condition can be illustrated below.*

A cantilevered beam under boundary conditions	A cantilevered beam under initial conditions
 	
Problem: We want to know what is the vibration response occurs under the *boundary conditions*: $x(0) = 4m$ & $x(10) = 7m$ (i.e. we displace the tip of the beam by 4cm upward and displace the middle of the beam by 7cm)	**Problem:** We want to know what is the vibration response occurs under the *initial conditions*: $x(0) = 4m$ & $x'(0) = 8m/s$ (i.e. we displace the tip of the beam by 4m upward and impact the tip simultaneously to cause it moving with initial velocity)

Note: For 1st order differential equation, the condition can be treated as initial/ boundary condition. For higher order differential equation, the distinction becomes obvious.