FOURIER SERIES & ITS APPLICATION

WEEK 10: FOURIER SERIES & ITS APPLICATION

10.1 INTRODUCTION

In engineering mathematic 1, you have learned various types of series such as

(i) Power Series, $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$

(ii) Frobenius Series, $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r}$

which models the polynomial function by the summation of infinite number of quantities/ terms. Previously, we have demonstrated that series is useful in solving engineering problem such as ODE, where we have successfully applied Power series and Frobenius series methods to solve 2nd order variable coefficient linear homogenous ODE with ordinary point and regular singular point respectively.

In engineering mathematic 2, you will learn a new series called Fourier series. Fourier series can be used to model any types of periodic function. It is useful to solve nonhomogenous ODE with periodic excitation.

(i) Fourier Series, $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega x + b_n \sin n\omega x)$

10.2 PERIODIC VS NON-PERIODIC FUNCTIONS

Before we continue, students should have the basic understanding on the definition of periodic & non-periodic functions.

Periodic Function	**Non-Periodic Function**
- Periodic function is a function that *repeats* its values at regular intervals (i.e. period *does exist*)	- Non-periodic function is a function that *does not repeat* its values at regular intervals (i.e. period *doesn't exist*) - Also known as aperiodic function
In mathematic, periodic function $f(t)$ is given: $f(t) = f(t + np)$	In mathematic, aperiodic function $f(t)$ is given: $f(t) \neq f(t + np)$
Where $n = integer = 1,2,\ldots$ $p = period\ (unit: s)$ $L = half\ of\ the\ period = \frac{p}{2}$ In simple, period is the *time taken* to move from its *starting point* and return to the *original point*	

Graphical representation of a periodic signal is illustrated in Figure 10.1. The signal repeats itself after a certain period, p, where $f(t) = f(t + np)$ is valid.

Example: constant, sine function, cosine function, tangent function, etc.

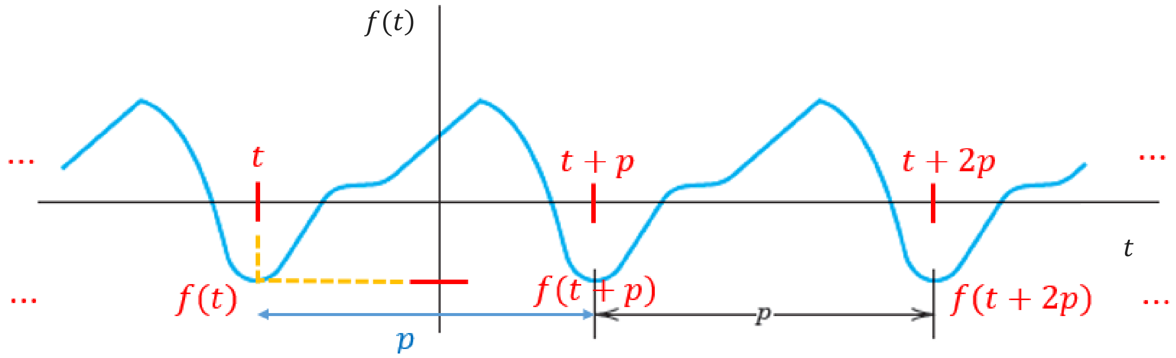


Figure 10.1: An example of periodic signal

Graphical representation of a non-periodic signal is illustrated in Figure 10.2. The signal does not repeats itself after certain period, p, where $f(t) \neq f(t + np)$.

Example: x, x^2, x^3, e^x, $\ln x$, etc.

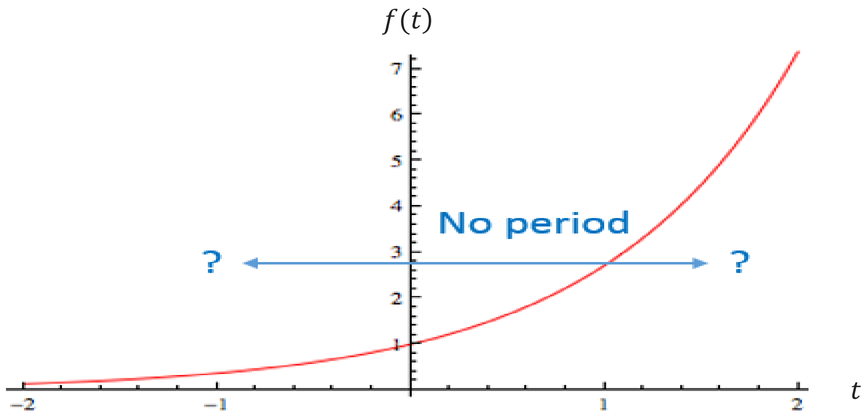


Figure 10.2: An example of non-periodic signal

Note: Student should be able to retrieve all important parameters from a periodic signal as follow.

(i) $p = period\ (unit{:}\ s)$

(ii) $L = half\ of\ the\ period = \frac{p}{2}$

(iii) $f = frequency\ (unit{:}\ Hz) = \frac{1}{p}$

(iv) $\omega = angular\ frequency\ \left(unit{:}\frac{rad}{s}\right) = 2\pi f = \frac{2\pi}{p}$

(v) The mathematical representation of a periodic signal, $f(t) = f(t + np)$

Example: Retrieve the $p, L, f\ \&\ \omega$ from the following function and write the mathematical representation of the periodic function.

(i) $5\sin(t)$

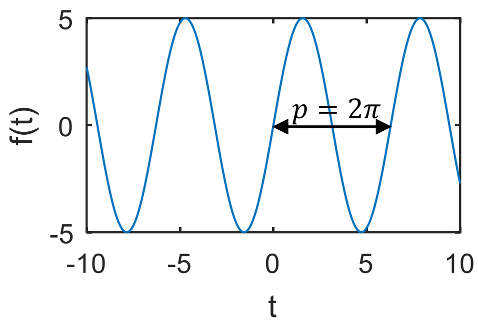
(ii) $10\cos(\frac{\pi}{5}t)$

(iii) 7

Solution:

(i) The general formula for the sine wave is $A\sin(\omega t)$, where $A =$amplitude, $\omega =$angular frequency.

- Thus, $5\sin(t)$ has $A = 5$and $\omega = 1$
- $frequency, f = \frac{\omega}{2\pi} = \frac{1}{2\pi}$
- $period, p = \frac{1}{f} = 2\pi$
- $half\ of\ the\ period, L = \frac{p}{2} = \pi$
- Mathematical representation of the periodic function, $f(t) = f(t + 2\pi n)$ where $5\sin(t) = 5\sin(2\pi n + t)$
- Also known as the periodic function with period of $2\pi\ seconds$



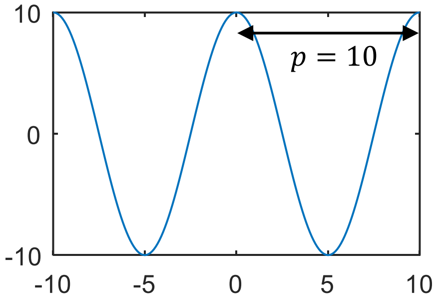
Observation:

Repeating itself over finite period, $p = 2\pi$

(e.g. $f(0) = f(2\pi)$)

(ii) The general formula for the cosine wave is $A\cos(\omega t)$, where $A =$amplitude, $\omega =$angular frequency.

- Thus, $10\cos(\frac{\pi}{5}t)$ has $A = 10$ and $\omega = \frac{\pi}{5}$
- $frequency, f = \frac{(\pi/5)}{2\pi} = \frac{1}{10}$
- $period, p = \frac{1}{f} = 10$
- $half\ of\ the\ period, L = \frac{p}{2} = 5$
- Mathematic representation of the periodic function, $f(t) = f(t + 10n)$ where $10\cos(\frac{\pi}{5}t) = 5\sin(2\pi n + t)$
- Also known as the periodic function with period of $10\ seconds$



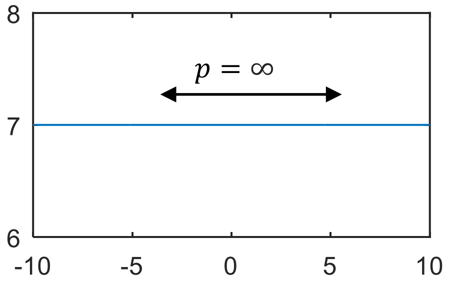
Observation:

Repeating itself over finite period, $p = 10$

(e.g. $f(0) = f(10)$)

(iii) The general formula for the constant is A, where A =amplitude.

- Thus, $A = 7$
- 7 is a periodic function with mathematical representation of, $f(t) = f(t + \infty)$ where $period, p = \infty$
- $half\ of\ the\ period, L = \frac{p}{2} = \infty$
- $frequency, f = \frac{1}{p} = 0$
- angular frequency, $\omega = 2\pi f = 0$



Observation:

Repeating itself over undefined period, $p = \infty$ (e.g. $f(0) = f(\infty)$)

Exercise 1: Check if $4\sin(15t) = 4\sin(15t + 2\pi n)$ is correct. If not correct, rewrite the mathematical representation of the periodic function.

Exercise 2: Give an example of periodic function with period of $\pi\ seconds$

Exercise 3: Give an example of periodic function with angular frequency of $\frac{\pi}{5}\ rad/s$

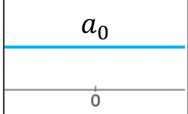
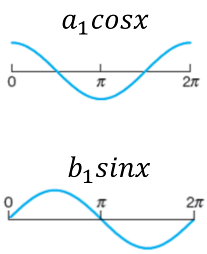
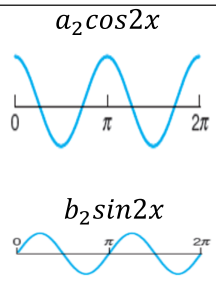
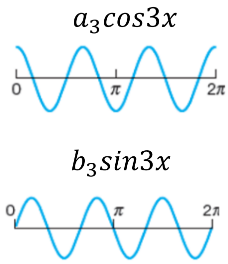
10.3 TRIGONOMETRIC SERIES AND FOURIER SERIES

Trigonometric series is a series of the following form:

$$f(x) = \underbrace{a_0}_{arbitrary\ constant} + \underbrace{\sum_{n=1}^{\infty} (a_n \cos n\omega x + b_n \sin n\omega x)}_{sinusoidal\ functions}$$

,which is the summation of an arbitrary constant and linear superposition of infinite number of sinusoidal functions (i.e. function with a cosine function and a sine function). More information about the sinusoidal function can be found in Appendix 10.1.

Graphical representation of the RHS of the Trigonometric series with fundamental angular frequency, $\omega = 1$.

Arbitrary constant, a_0	1^{st} term of the sinusoidal function, $SF_1 =$ $a_1 \cos x + b_1 \sin x$	2^{nd} term of the sinusoidal function, $SF_2 =$ $a_2 \cos 2x + b_2 \sin 2x$	3^{rd} term of the sinusoidal function, $SF_3 =$ $a_3 \cos 3x + b_3 \sin 3x$	And so on…
				+ …

Let the coefficient a_n & b_n to be non-zero arbitrary coefficient. We can simplify it to be

$$f(x) = a_0 + SF_1 + SF_2 + \cdots + SF_n$$

where SF_n denoted the n^{th} term of the sinusoidal function.

For example:

$f(x) = 5 + (6\,cosx + 3sinx) + (10cos2x + 1.5sin2x) + (8cos3x + 4sin3x) + \cdots + SF_n$

where $a_0 = 5$

$SF_1 = 6\,cosx + 3sinx$

$SF_2 = 10cos2x + 1.5sin2x$

$SF_3 = 8cos3x + 4sin3x$

$SF_n = a_n \cos nx + b_n \sin nx\,, n = 1,2, \dots, \infty$

Fourier series is extended from the previous Trigonometric series, where all the unknown coefficient a_n & b_n can be found by the *Euler's formulae* below. Note that Fourier series is applicable for any periodic function with arbitrary period of $p = 2L$.

$$f(x) = \underbrace{a_0}_{arbitrary\ constant} + \underbrace{\sum_{n=1}^{\infty} (a_n \cos n\omega x + b_n \sin n\omega x)}_{sinusoidal\ functions}$$

where $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\,cos\,n\omega x\ \,dx$

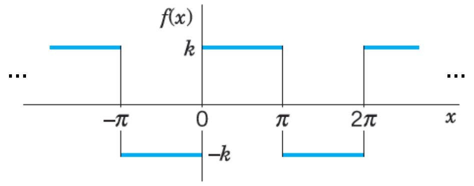
$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\,sin\,n\omega x\ \,dx$

$\omega = \frac{2\pi}{p} = \frac{\pi}{L}$

Extra Info: Check appendices 11.2 & 11.3 to understand how to derive the Euler's formulae as well as the convergence of the Fourier series

Fourier series is named in honour of Jean-Baptiste Joseph Fourier (1768-1830), who found that the *trigonometric series can be used to represent a periodic function*. In fact, a complicated periodic signal is merely a linear superposition of multiple sine and cosine waves.

Example 10.3.1:

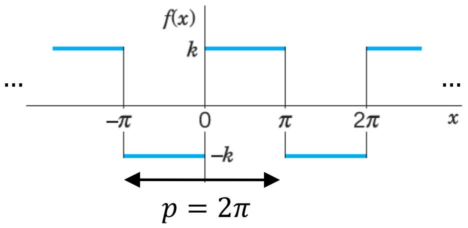


(i) Is the signal above a periodic signal or non-periodic signal?

Solution (i): The signal repeats itself over finite period, thus it is a periodic signal.

(ii) If it is a periodic signal, identify its period. Hence, write its function.

Solution (ii):



- *The time function of the rectangular wave:*
- $f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0 \\ 1 & \text{if } 0 < x < \pi \end{cases}$ and $f(x) = f(x + n(2\pi))$ *where* $n = 1,2, \dots, \infty$

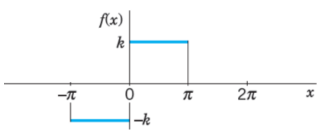
Precaution: This information "$f(x) = f(x + n(2\pi))$" must be provided to indicate that it is a periodic signal.

Example 10.3.2:

Find the Fourier series of the following signal.

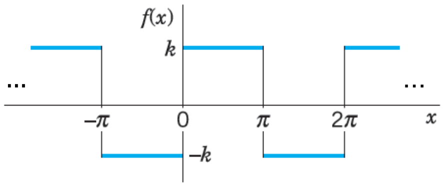
(i) $f(x) = \begin{cases} -k & \text{if } -\pi < x < 0 \\ k & \text{if } 0 < x < \pi \end{cases}$

Solution (i): The signal does not repeat itself, thus it is a non-periodic signal. Fourier series cannot be applied to non-periodic signal.



(ii) $f(x) = \begin{cases} -k & \text{if } -\pi < x < 0 \\ k & \text{if } 0 < x < \pi \end{cases}$ & $f(x) = f(x + n(2\pi))$ *where* $n = 1,2,\dots,\infty$

Solution (ii): The signal repeat itself over finite period, $p = 2\pi$, thus it is a periodic signal. Fourier series can be applied to periodic signal.



Fourier Series expression:

$$f(x) = \underbrace{a_0}_{arbitrary\ constant} + \underbrace{\sum_{n=1}^{\infty} (a_n \cos n\omega x + b_n \sin n\omega x)}_{sinusoidal\ functions}$$

where $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos n\omega x\;dx$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin n\omega x\;dx$

Step 1: Retrieve all important parameters from the periodic signals.

Period, $p = 2\pi$;

Half of the Period, $L = \pi$;

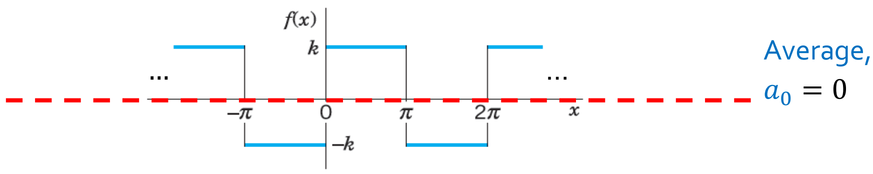
Frequency, $f = \frac{1}{2\pi}\,Hz$

Angular frequency, $\omega = \frac{2\pi}{p} = 1\,rad/s$

Step 2: Solve the coefficient a_0

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$
$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-k)dx + \int_{0}^{\pi}(k)dx\right]$$
$$= \frac{1}{2\pi}[-kx]_{-\pi}^{0} + \frac{1}{2\pi}[kx]_{0}^{\pi}$$
$$= \frac{1}{2\pi}[0-(-k)(-\pi)] + \frac{1}{2\pi}[(k)(\pi)-0]$$
$$= 0$$

***Comment**: a_0 indicates the average of the periodic signal.*



Step 3: Solve the coefficient a_n

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos n\omega x \ dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \ dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -k\cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} k\cos nx\, dx$$

$$= \frac{1}{\pi}\left[-k\frac{sinnx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[k\frac{sinnx}{n}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[0-(-k)(\frac{sinn(-\pi)}{n})\right] + \frac{1}{\pi}\left[(k)\left(\frac{sinn(\pi)}{n}\right)-0\right]$$

$$= \frac{1}{\pi}\left[(k)\left(\frac{sinn(-\pi)}{n}\right)+(k)\left(\frac{sinn(\pi)}{n}\right)\right] \qquad \textit{[Hint: } sinn(-\pi) = -sinn(\pi)\textit{]}$$

$$= 0$$

Step 4: Solve the coefficient b_n

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin n\omega x \ dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \ dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -k\sin nx\, dx + \frac{1}{2\pi}\int_{0}^{\pi} k\sin nx\, dx$$

$$= \frac{1}{\pi}\left[-k\frac{cosnx}{-n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[k\frac{cosnx}{-n}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{k}{n}-(-k)(\frac{cosn(-\pi)}{-n})\right] + \frac{1}{\pi}\left[(k)\left(\frac{cosn(\pi)}{-n}\right)-\frac{k}{-n}\right]$$

$$= \frac{1}{\pi}\left[\frac{2k}{n}-(k)\left(\frac{cosn(-\pi)}{n}\right)+(k)\left(\frac{cosn(\pi)}{-n}\right)\right] \quad \textit{[Hint: } cosn(-\pi) = cosn(\pi)\textit{]}$$

$$= \frac{2k}{n\pi}[1-cos(n\pi)] \ \ where\ n = 1,2,3,\ldots$$

$$b_n = \begin{cases} \frac{2k}{n\pi}[1-(-1)] & for\ odd\ n \\ \frac{2k}{n\pi}[1-(1)] & for\ even\ n \end{cases} \qquad \textit{[Hint: } \cos n\pi = \begin{cases} -1 & for\ odd\ n \\ 1 & for\ even\ n \end{cases} = (-1)^n\textit{]}$$

$$b_n = \begin{cases} \frac{4k}{n\pi} & for\ odd\ n \\ 0 & for\ even\ n \end{cases}$$

$$\therefore b_1 = \frac{4k}{\pi}, \quad b_2 = 0, \quad b_3 = \frac{4k}{3\pi}, \quad b_4 = 0, \quad b_4 = \frac{4k}{5\pi}, \quad \ldots$$

Step 5: Express the signal in the form of Fourier series.

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos n\omega x + b_n \sin n\omega x)$$

where $a_0 = 0,$ $\qquad a_n = 0,$ $\qquad b_n = \begin{cases} \frac{4k}{n\pi} & for\ odd\ n \\ 0 & for\ even\ n \end{cases}$

$$f(x) = a_0 + (a_1\cos x + b_1\sin x) + (a_2\cos 2x + b_2\sin 2x) + (a_3\cos 3x + b_3 \sin 3x) + \\ +(a_4\cos 4x + b_4\sin 4x) + \cdots$$

$$\therefore\ f(x) = \frac{4k}{\pi}\sin x + \frac{4k}{3\pi}\sin 3x + \frac{4k}{5\pi}\sin 5x + \frac{4k}{7\pi}\sin 7x \ldots$$

Comment: The complicated square function with period of $p = 2\pi$ is the linear superposition result of multiple sine waves with odd frequencies.

10.4 APPLICATION OF FOURIER SERIES #1: PLOTTING A PERIODIC FUNCTION

Previously, we demonstrated that a periodical square function can be represented in the form of Fourier series as follows.

$$f(x) = \begin{cases} -k & \text{if } -\pi < x < 0 \\ k & \text{if } 0 < x < \pi \end{cases} \quad \text{and } f(x) = f(x + n(2\pi)) \text{ where } n = 1,2,\dots,\infty$$

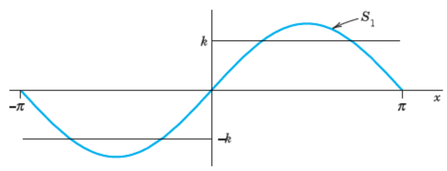
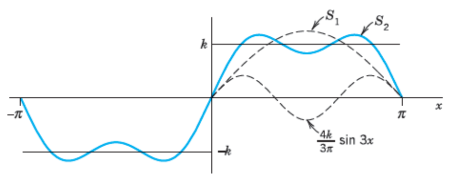
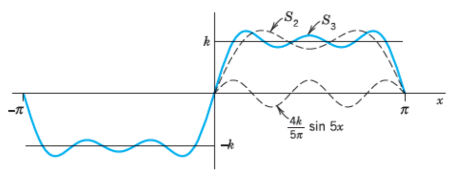
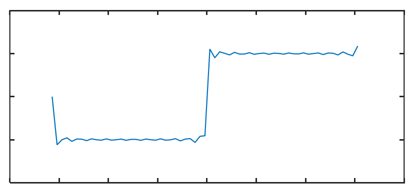
[Periodic square wave]

$$\Downarrow$$

$$f(x) = \frac{4k}{\pi}\sin x + \frac{4k}{3\pi}\sin 3x + \frac{4k}{5\pi}\sin 5x + \frac{4k}{7\pi}\sin 7x + \cdots$$

[Fourier series]

In fact, we can use the Fourier series expression to plot the periodic square function.

Partial Summation, S_n	**Approximation to $f(x)$** $f(x) \approx S_n$	**Graphical representation**
S_1	$S_1 = \frac{4k}{\pi}\sin 1x$	 Poor approximation to rectangular wave
S_2	$S_2 = \frac{4k}{\pi}\sin 1x + \frac{4k}{3\pi}\sin 3x$	
S_3	$S_3 = \frac{4k}{\pi}\sin 1x + \frac{4k}{3\pi}\sin 3x + \frac{4k}{5\pi}\sin 5x$	
⋮	⋮	⋮
S_{20}	$S_{20} = \frac{4k}{\pi}\sin 1x + \frac{4k}{3\pi}\sin 3x + \frac{4k}{5\pi}\sin 5x + \cdots + \frac{4k}{39\pi}\sin 39x$	 Good approximation to rectangular wave

Comment: In common practice, *partial summation of 20 terms* is used to give a *good approximation* of a periodic signal. If higher accuracy is needed, the number of terms will be increased.

10.5 APPLICATION OF FOURIER SERIES #2: DECOMPOSE A PERIODIC FUNCTION INTO MULTIPLE SINUSOIDAL WAVES WITH VARIOUS FREQUENCIES

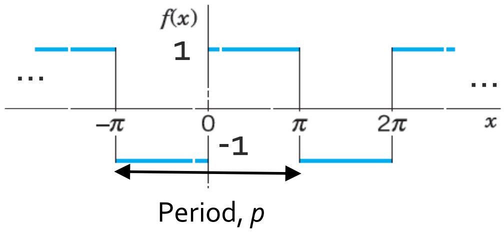
The Fourier series of a rectangular wave with amplitude = 1 is given as follows.

$$f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0 \\ 1 & \text{if } 0 < x < \pi \end{cases} \quad \text{and } f(x) = f(x + n(2\pi)) \text{ where } n = 1,2,\ldots,\infty$$

$$\Downarrow$$

$$f(x) = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x + \frac{4}{7\pi}\sin 7x \ldots$$

[Fourier series]



Question: *What is the frequency contaminated in the rectangular wave with amplitude = 1?*

Solution: Based on the Fourier series result above, the Fourier series decomposed the rectangular wave into linear superposition of multiple sine functions with <u>*odd-integer harmonic angular frequency.*</u> *In other words, the rectangular wave contains infinity number of* <u>*odd-integer harmonic angular frequency.*</u>

(i.e. First or fundamental harmonic (1x) frequency = $\omega_1 = 1rads^{-1}$ *; Second harmonic (2x) frequency =* $\omega_2 = 3rads^{-1}$ *, ...)*

<table>
<tr><th>Time Domain Perspective of
$f(x) \approx S_n$</th><th>Frequency Domain Perspective of
$f(x) = rectangular\ wave$</th></tr>
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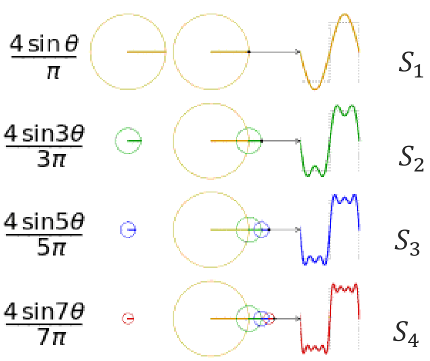

Comment: S_4 indicates the 4 terms summation result, i.e. the approximation of $f(x)$ by partial summation.
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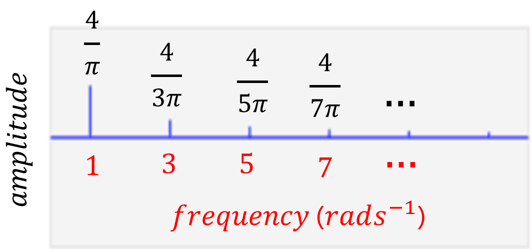
The frequency components of the $f(x)$ can be represented by the following figure.



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Graphical representation of the time & frequency domains perspective of $f(x)$ and the decomposed components of the Fourier series (i.e. $SF_1, SF_2, SF_3, \ldots$).

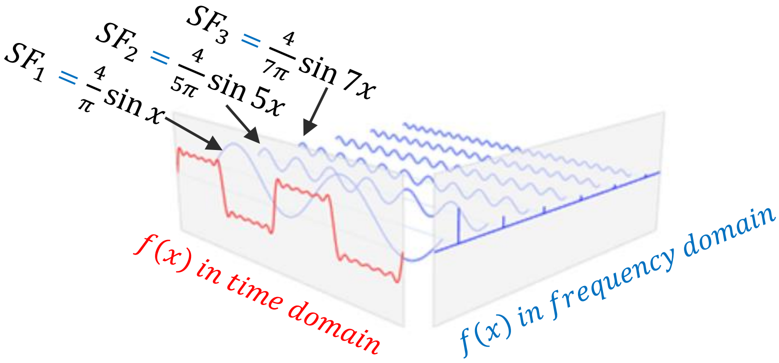


Figure 10.5: Time and Frequency domains of a periodical rectangular wave

Background: Joseph Fourier (1822) states that a time signal can be decomposed not only in time domain in terms of a sequence of sinusoidal waves, but also in frequency domain as well in terms of different frequency components. This idea makes a huge impacts and give innovation of many inclusive ideas in various engineering applications including vibration analysis, electrical analysis, acoustic analysis, image processing such as image compression, signal processing, quantum mechanics, etc. The first intention of Fourier's work is to solve the heat diffusion or transient heat conduction model by using the Fourier series approach. Later, his work directly influenced and inspired others to use similar approach to describe other dynamic physical systems.

10.6 APPLICATION OF FOURIER SERIES #3: TO OBTAIN FINITE RESULT OF A SERIES

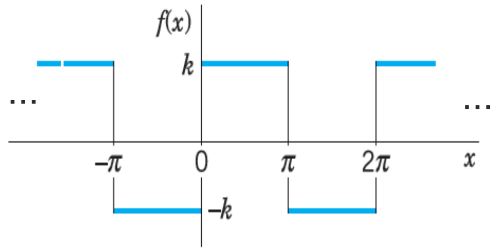
It is difficult to determine the result of a series, e.g. what is the result of the series below?

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=? \qquad \text{(Problem 1)}$$

Or in some cases, you might wonder how a famous series was formed or proven, e.g.

Leibniz series: $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=\frac{\pi}{4}$ (Problem 2)

In this section, we will demonstrated how to use Fourier series to find/ prove the finite result of a particular series. For example, the Fourier series of the periodical rectangular wave is given.



$$\therefore f(x)=\frac{4k}{\pi}\sin x+\frac{4k}{3\pi}\sin 3x+\frac{4k}{5\pi}\sin 5x+\frac{4k}{7\pi}\sin 7x+\ldots$$

Try to substitute various x to the Fourier series above:

<u>First Attempt, let x=π/4:</u>

x	LHS OF $f(x)$	RHS OF $f(x)$
$\frac{\pi}{4}$	$f\left(\frac{\pi}{4}\right)=k$	$\frac{4k}{\pi}\sin\frac{\pi}{4}+\frac{4k}{3\pi}\sin\left(3\frac{\pi}{4}\right)+\frac{4k}{5\pi}\sin\left(5\frac{\pi}{4}\right)+\frac{4k}{7\pi}\sin\left(7\frac{\pi}{4}\right)+\cdots$ $=\frac{4k}{\pi}(0.707)+\frac{4k}{3\pi}(0.707)+\frac{4k}{5\pi}(-0.707)+\frac{4k}{7\pi}(-0.707)+\cdots$
	LHS = RHS $k=\frac{4k}{\pi}(0.707)+\frac{4k}{3\pi}(0.707)+\frac{4k}{5\pi}(-0.707)+\frac{4k}{7\pi}(-0.707)+\cdots$ We obtain a new series, where $1=\frac{4}{\pi}(0.707)+\frac{4}{3\pi}(0.707)+\frac{4}{5\pi}(-0.707)+\frac{4}{7\pi}(-0.707)+\cdots$ Rearrange, $\frac{\pi}{4}=(0.707)+\frac{1}{3}(0.707)+\frac{1}{5}(-0.707)+\frac{1}{7}(-0.707)+\cdots$	

Second attempt, let $x=\pi/2$:

x	LHS OF $f(x)$	RHS OF $f(x)$
$\frac{\pi}{2}$	$f\left(\frac{\pi}{2}\right)=k$	$\frac{4k}{\pi}\sin\frac{\pi}{2}+\frac{4k}{3\pi}\sin\left(3\frac{\pi}{2}\right)+\frac{4k}{5\pi}\sin\left(5\frac{\pi}{2}\right)+\frac{4k}{7\pi}\sin\left(7\frac{\pi}{2}\right)+\cdots$
	LHS = RHS $k=\frac{4k}{\pi}(1)+\frac{4k}{3\pi}(-1)+\frac{4k}{5\pi}(1)+\frac{4k}{7\pi}(-1)+\cdots$ We obtain a new series, where $1=\frac{4}{\pi}-\frac{4}{3\pi}+\frac{4}{5\pi}-\frac{4}{7\pi}+\cdots$ Rearrange it, we obtain $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$	

Third attempt, let $x=9\pi/5$:

x	LHS OF $f(x)$	RHS OF $f(x)$
$\frac{9\pi}{5}$	$f\left(\frac{9\pi}{5}\right)=-k$	$\frac{4k}{\pi}\sin\frac{9\pi}{5}+\frac{4k}{3\pi}\sin\left(3\frac{9\pi}{5}\right)+\frac{4k}{5\pi}\sin\left(5\frac{9\pi}{5}\right)+\frac{4k}{7\pi}\sin\left(7\frac{9\pi}{5}\right)+\cdots$
	LHS = RHS $-k=\frac{4k}{\pi}(-0.588)+\frac{4k}{3\pi}(-0.951)+\frac{4k}{5\pi}(0)+\frac{4k}{7\pi}(0.951)+\cdots$ We obtain a new series, where $-1=\frac{4}{\pi}(-0.588)+\frac{4}{3\pi}(-0.951)+\frac{4}{7\pi}(0.951)+\cdots$ Rearrange it, we obtain $\frac{\pi}{4}=(0.588)+\frac{1}{3}(0.951)-\frac{1}{7}(0.951)+\cdots$	

Think: You have tried 3 attempts and produce three new series from the Fourier series. In fact, you can produce infinite types of series based on the Fourier series result. Tried to link the attempt to the Problem 1 & Problem 2.

Solution to Problem 1: By selecting appropriate x such as the one in 2nd attempt, we obtain

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}$$

Solution to Problem 2: LHS of Leibniz series: $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}$ (Proven)

Extra info: Leibniz series is named after Gottfried Leibniz who succeed to discover π in series format, i.e.

$\pi=4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=4-\frac{4}{3}+\frac{4}{5}-\frac{4}{7}+\cdots$. So far, there is no series expression yet for the imaginary number, i. Perhaps anyone here can express it using this approach? Think about discover it, and one day you might put your big name to those unnamed series.

10.7 FOURIER COSINE SERIES & FOURIER SINE SERIES

The equations for the Fourier Cosine Series & Fourier Sine Series are given below:

(i) Fourier Cosine Series:

$$f_c(x) = \underbrace{a_0}_{arbitrary\ constant} + \underbrace{\sum_{n=1}^{\infty}(a_n \cos n\omega x)}_{sinusoidal\ functions}$$

where $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos n\omega x\ dx$

$\omega = \frac{2\pi}{p} = \frac{\pi}{L}$

(ii) Fourier Sine Series:

$$f_s(x) = \underbrace{\sum_{n=1}^{\infty}(b_n \sin n\omega x)}_{sinusoidal\ function}$$

where $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin n\omega x\ dx$

Note that we will obtain Fourier series by the summation of the Fourier Cosine series and Fourier Sine series. In other words, Fourier series is formed by Fourier Cosine series and Fourier Sine series.

$$Fourier\ series, f(x) = Fourier\ Cosine\ series, f_c(x) + Fourier\ Sine\ series, f_s(x)$$

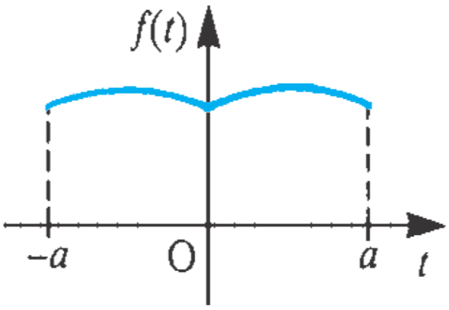
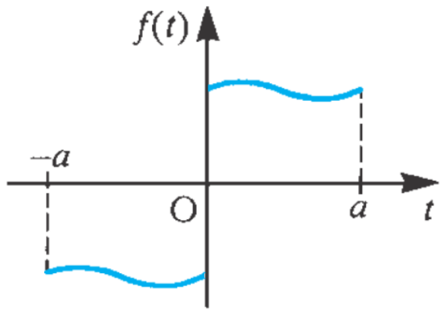
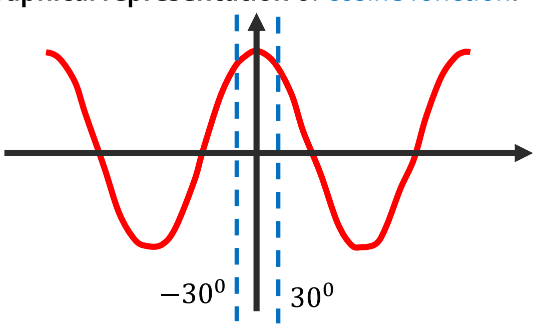
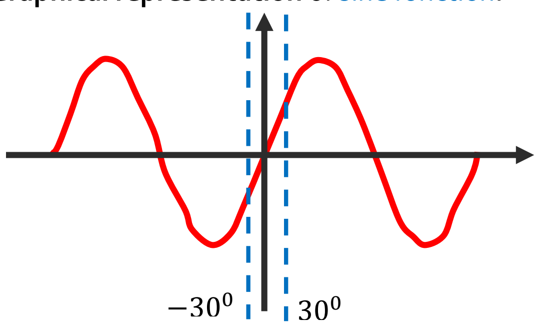
Note 1: There is one important characteristic that is possessed by the Fourier Cosine series and Fourier Sine series that we must understand, i.e. Odd and Even function, which will be discussed in the next section.

Note 2: $Fourier\ Cosine\ series$, $f_c(x)$ is an even function

Note 3: $Fourier\ Sine\ series$, $f_s(x)$ is an odd function

10.8 EVEN FUNCTION AND ODD FUNCTION

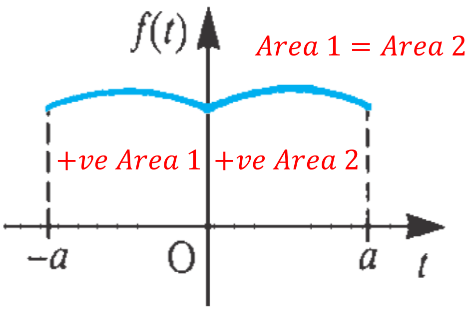
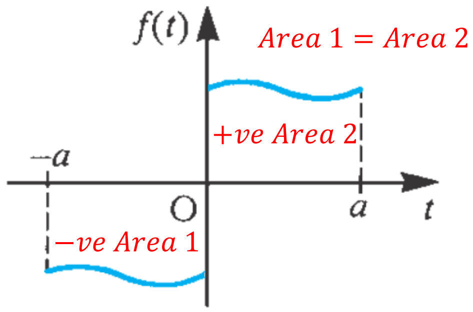
The definitions of the even and odd functions are given below:

<table>
<tr><th>EVEN FUNCTION</th><th>ODD FUNCTION</th></tr>
<tr><td>Mathematical definition:
$$f(-t) = f(t)$$</td><td>Mathematical definition:
$$f(-t) = -f(t)$$</td></tr>
<tr><td>Graphical representation of $f(t)$:

</td><td>Graphical representation of $f(t)$:

</td></tr>
<tr><td>Example: Cosine function is an even function because it satisfies
$$f(-t) = f(t)$$
where
$$cos(-t) = cos(t)$$
Eg.
$$cos(-30^0) = cos(30^0) = 0.866$$</td><td>Example: Sine function is an odd function because it satisfies
$$f(-t) = -f(t)$$
where
$$sin(-t) = -sin(t)$$
Eg.
$$sin(-30^0) = -sin(30^0) = -0.5$$</td></tr>
<tr><td>Graphical representation of cosine function:

</td><td>Graphical representation of sine function:

</td></tr>
</table>

Observation 1: For even function, the *y axis acts like a mirror* to copy data from $+t$ to $-t$ domain.
Observation 2: For odd function, the *y axis acts like an upside-down mirror* to copy data from $+t$ to $-t$ domain in an upside-down manner.

Exercise: Identify if tangent function, Fourier Sine series and Fourier Cosine series are even or odd function by using the definition above.

Important characteristics of even and odd functions:

Total area under the even function graph	**Total area under the odd function graph**
Total area from $-a$ to a for even function:  Total area, $\int_{-a}^{a} f(t)\,dt = Area\ 1 + Area\ 2$ $= 2\ x\ Area\ 2$ **Characteristic of even function:** $\int_{-a}^{a} f(t)\,dt = 2\int_{0}^{a} f(t)\,dt$	Total area from $-a$ to a for odd function:  Total area, $\int_{-a}^{a} f(t)\,dt = Area\ 1 + Area\ 2$ $= 0$ **Characteristic of odd function:** $\int_{-a}^{a} f(t)\,dt = 0$

By learning the characteristic of the even and odd functions, we can simplify the calculation of Fourier series in some cases. For example:

(i) ***<u>If function $f(t)$ is not an even or odd function,</u>***

Fourier series cannot be simplified to Fourier Cosine series or Fourier Sine series alone. It is a combination of both of them.

$$\text{Fourier series}, f(t) = \text{Fourier Cosine series}, f_c(t) + \text{Fourier Sine series}, f_s(t)$$
$$= a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$$

(ii) ***<u>If function $f(t)$ is an even function,</u>***

Fourier series can be simplified to Fourier Cosine series

$$\text{Fourier series}, f(t) = \text{Fourier Cosine series}, f_c(t)$$
$$= a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t)$$

(iii) ***<u>If function $f(t)$ is an odd function,</u>***

Fourier series can be simplified to Fourier Sine series

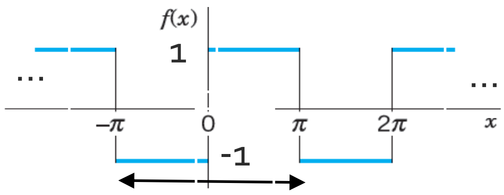
$$\text{Fourier series}, f(t) = \text{Fourier Sine series}, f_s(t)$$
$$= \sum_{n=1}^{\infty}(b_n \sin n\omega t)$$

Note 1: Approach (i) is <u>time consuming</u>, followed by (ii) and (iii), as approach (i) needs to calculate 3 unknowns (a_0, a_n & b_n), while (ii)-(2 unknowns a_0&a_n) & (iii)-(1 unknown b_n).

Note 2: This means that if we able to <u>identify whether a function is an odd or even function</u>. We can *use the Fourier Cosine series or Fourier Sine series to make the calculation easier and faster*.

Example 1: Previously we use Fourier series approach to find the series of a periodic rectangular wave. By learning the characteristic of the odd and even function, we can simplify the calculation using Fourier Cosine series or Fourier Sine series approaches as follows.

$f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0 \\ 1 & \text{if } 0 < x < \pi \end{cases}$ and $f(x) = f(x + n(2\pi))$ $where\ n = 1,2, \ldots, \infty$



Period, *p*

[Fourier series approach]

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega x + b_n \sin n\omega x)$$

[Odd Function- Fourier Sine Series]

Comment: Based on the figure, $f(x)$ is an odd function because the f(x) axis acts like an *upside-down mirror* to copy data from $+t$ to $-t$ domain in an upside-down manner.

Thus,

$$f(x) = \mathit{Fourier\ Sine\ series} = \sum_{n=1}^{\infty} (b_n \sin n\omega x)$$

where $b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin n\omega x \; dx = \begin{cases} \frac{4k}{n\pi} & for\ odd\ n \\ 0 & for\ even\ n \end{cases}$

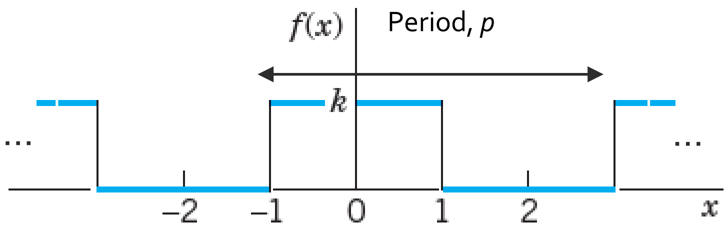
[Same answer as previous]

$$f(x) = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x + \frac{4}{7\pi}\sin 7x \ldots$$

Example 2:

Find the Fourier series of the function

$$x = \begin{cases} 0 & if \quad -2 < x < -1 \\ k & if \quad -1 < x < 1 \\ 0 & if \quad 1 < x < 2 \end{cases} \quad \& \; f(x) = f(x + n(4)) \text{ where } n = 1,2,3, \ldots$$



Step 1: Extract all the important information from the figure.

Period, $p = 4$; *Half period*, $L = 2$

Angular frequency, $\omega = \frac{2\pi}{4} = \frac{\pi}{2}$; *Frequency*, $f = \frac{1}{4}$

Step 2: Check if the function is solely an odd or even function or neither of them.

Based on the figure, $f(x)$ is an even function because the $f(x)$ axis acts like a mirror to copy data from $+t$ to $-t$ domain.

Step 3: Fourier Cosine series

$$\textit{Fourier series}, f(x) = \textit{Fourier Cosine series}, f_c(x)$$
$$= a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega x)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx = \frac{1}{4}\int_{-2}^{2} f(x)\, dx$$
$$= \frac{1}{4}\left(\int_{-2}^{-1} 0\, dx + \int_{-1}^{1} k\, dx + \int_{1}^{2} 0\, dx\right)$$
$$= \frac{k}{2}$$

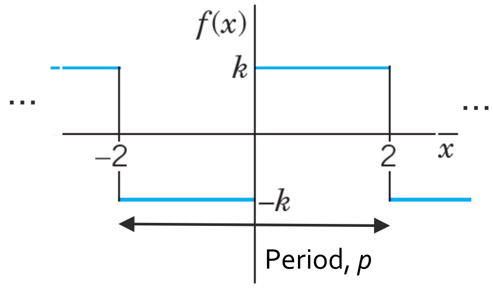
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos n\omega x\, dx = \frac{1}{2}\int_{-2}^{2} f(x) \cos n\frac{\pi}{2} x\, dx$$
$$= \frac{1}{2}\left(\int_{-1}^{1} k \cos n\frac{\pi}{2} x\, dx\right)$$
$$= \frac{k}{2}\left[\frac{\sin n\frac{\pi}{2}x}{n\frac{\pi}{2}}\right]_{-1}^{1} = \frac{k}{n\pi}\left(\sin n\frac{\pi}{2} - \sin\left(-n\frac{\pi}{2}\right)\right) = \frac{2k}{n\pi} \sin\left(n\frac{\pi}{2}\right)$$

$$a_n = \begin{cases} 2k/n\pi & if \; n = 1, 5, 9, \ldots \\ -2k/n\pi & if \; n = 3, 7, 11, \ldots \end{cases}$$

Example 3:

Find the Fourier series of the function

$$f(x) = \begin{cases} -k & \text{if } -2 < x < 0 \\ k & \text{if } 0 < x < 2 \end{cases} \quad \& \ f(x) = f(x + n(4)) \text{ where } n = 1,2,3, \ldots$$



Solution:

Step 1: Extract all the important information from the figure.

Period, $p = 4$; *Half period*, $L = 2$

Angular frequency, $\omega = \frac{2\pi}{4} = \frac{\pi}{2}$; *Frequency*, $f = \frac{1}{4}$

Step 2: Check if the function is solely an odd or even functions or neither of them.

Based on the figure, $f(x)$ is an odd function because the f(x) axis acts like a mirror to copy data from $+t$ to $-t$ domain.

Step 3: Fourier Sine series

$$\textit{Fourier series}, f(t) = \textit{Fourier Sine series}, f_s(t) = \sum_{n=1}^{\infty}(b_n \sin n\omega t)$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x) \sin n\frac{\pi}{2}x \ dx \text{ ,where } n = 1,2,3, \ldots$$

$$= \frac{1}{2}\left(\int_{-2}^{0} -k \sin n\frac{\pi}{2}x \, dx + \int_{0}^{2} k \sin n\frac{\pi}{2}x \, dx\right)$$

$$= \frac{-k}{2}\left[\frac{-\cos n\frac{\pi}{2}x}{n\frac{\pi}{2}}\right]_{-2}^{0} + \frac{k}{2}\left[\frac{-\cos n\frac{\pi}{2}x}{n\frac{\pi}{2}}\right]_{0}^{2}$$

$$= \frac{-k}{n\pi}\left(-\cos 0 - (-\cos(-n\pi))\right) + \frac{k}{n\pi}\left(\cos(n\pi) - (-\cos 0)\right)$$

$$= \frac{2k}{n\pi}\left(1 - \cos(n\pi)\right)$$

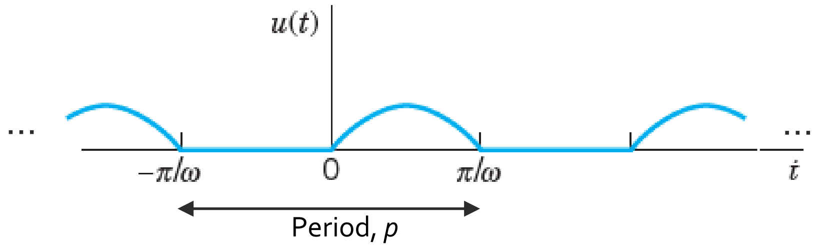
$$b_n = \begin{cases} 4k/n\pi & \text{if } n = \text{odd number} \\ 0 & \text{if } n = \text{even number} \end{cases}$$

$\therefore \ f(x) = \frac{4k}{\pi}\sin x + \frac{4k}{3\pi}\sin 3x + \frac{4k}{5\pi}\sin 5x + \frac{4k}{7\pi}\sin 7x + \cdots$,where $f(x)$ is valid for any interval $-\infty \le x \le \infty$

Example 4:

Find the Fourier series of the function

$$u(t) = \begin{cases} 0 & \text{if } -L < t < 0 \\ E \sin \omega t & \text{if } 0 < t < L \end{cases} \quad \& \; u(t) = u\left(t + n\left(\frac{2\pi}{\omega}\right)\right) \text{ where } n = 1,2,3, \ldots$$



Solution:

Step 1: Extract all the important information from the figure.

Period, $p = \frac{2\pi}{\omega}$; *Half period,* $L = \frac{\pi}{\omega}$

Angular frequency, $\omega = \frac{2\pi}{\left(\frac{2\pi}{\omega}\right)} = \omega$; *Frequency,* $f = \frac{1}{\left(\frac{2\pi}{\omega}\right)} = \frac{\omega}{2\pi}$

Step 2: Check if the function is solely an odd or even function or neither of them.

It is not an odd or even functions.

Step 3: Fourier Series

$u(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega\, t + b_n \sin n\omega\, t\right)$

$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\, dt = \frac{\omega}{2\pi}\int_0^L E \sin\omega\, t\, dt = \frac{\omega}{2\pi}\left[\frac{-E\cos\omega t}{\omega}\right]_0^L = \frac{\omega}{2\pi}\left(\frac{-E\cos\pi}{\omega} - \frac{-E}{\omega}\right) = \frac{E}{\pi}$

$a_n = \frac{1}{L}\int_{-L}^{L} f(t) \cos n\omega t\;\; dt$

$= \frac{\omega}{\pi}\left(\int_0^L E \sin\omega\, t \cos n\omega t\, dt\right)$

$= \frac{\omega E}{\pi}\frac{1}{2}\left(\int_0^L \sin(1+n)\omega t + \sin(1-n)\omega t\, dt\right)$

For $n = 1$, $a_1 = \frac{\omega E}{2\pi}\int_0^L \sin(2\omega t)\, dt = 0$

For $n > 1$,

$a_n = \frac{\omega E}{2\pi}\left(\left[\frac{-\cos(1+n)\omega t}{(1+n)\omega} - \frac{-\cos(1-n)\omega t}{(1-n)\omega}\right]_0^L\right) = \frac{\omega E}{2\pi}\left(\frac{-\cos(1+n)\pi}{(1+n)\omega} + \frac{\cos(1-n)\pi}{(1-n)\omega} - \left(\frac{-1}{(1+n)\omega} - \frac{-1}{(1-n)\omega}\right)\right)$

$a_2 = \frac{-2E}{3\pi}$, $a_3 = 0$, $a_4 = \frac{-2E}{(3x5)\pi}$, $a_5 = 0$, $a_6 = \frac{-2E}{(5x7)\pi}$, …

$b_n = \frac{1}{L}\int_{-L}^{L} f(t) \sin n\omega t\, dt = \frac{\omega}{\pi}\int_0^L E \sin\omega\, t \sin n\omega t\, dt$

$= \frac{\omega}{\pi}\int_0^L E\, \frac{\cos((1-n)\omega t) - \cos((1+n)\omega t)}{2}\, dt$

For $n = 1$

$$b_n = \frac{\omega}{\pi}\int_0^L E\,\frac{\cos(0)-\cos((2)\omega t)}{2}\,dt = \frac{\omega E}{2\pi}\left[t - \frac{sin(2)\omega t}{(2)\omega}\right]_0^L = \frac{\omega E}{2\pi}\left(\frac{\pi}{\omega} - \frac{\sin(2)\pi}{(2)\omega} - \left(0 - \frac{0}{(2)\omega}\right)\right) = \frac{E}{2}$$

For $n > 1$

$$b_n = \frac{\omega E}{2\pi}\left(\left[\frac{sin(1-n)\omega t}{(1-n)\omega} - \frac{-sin(1+n)\omega t}{(1+n)\omega}\right]_0^L\right) = \frac{\omega E}{2\pi}\left(\frac{\sin(1-n)\pi}{(1-n)\omega} + \frac{\sin(1+n)\pi}{(1+n)\omega} - \left(\frac{0}{(1+n)\omega} - \frac{0}{(1-n)\omega}\right)\right)$$

$$= \frac{\omega E}{2\pi}\left(\frac{\sin(1-n)\pi}{(1-n)\omega} + \frac{\sin(1+n)\pi}{(1+n)\omega}\right)$$

$$b_2 = \frac{\omega E}{2\pi}\left(\frac{\sin(-1)\pi}{(-1)\omega} + \frac{\sin(3)\pi}{(3)\omega}\right) = 0$$

Since $\sin(1-n)\pi = \sin(1+n)\,\pi = 0$ for all $n > 1$

$b_n = 0$ for $n = 2,3,4,\ldots$

$$\therefore f(t) = \frac{E}{\pi} + \left(-\frac{2E}{3\pi}(cos2\omega t) - \frac{2E}{(3x5)\pi}(cos4\omega t) - \frac{2E}{(5x7)\pi}(cos6\omega t) + \cdots\right) + \frac{E}{2}(sin\omega t)$$

where $f(x)$ is valid for any interval $-\infty \leq x \leq \infty$

In summary,

Important remark: The main function of Fourier series is to decompose the signal into infinite sinusoidal wave with various frequencies.
The $f(x)$ is valid for any interval $-\infty \le x \le \infty$

Any periodic functions
(i.e. $f(x) = f(x+np)$;
$where\ n = 1,2,3, \dots and\ p = period)$

Fourier Series (arbitrary period of p=2L)
$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega x + b_n \sin n\omega x)$
where $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$;
$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos n\omega x\ dx$;
$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin n\omega x\,dx$;

If $f(x)$ = even function

Fourier Cosine Series
$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega x)$
where $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$;
$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos n\omega x\ dx$;

If $f(x)$ = odd function

Fourier Sine Series
$f(x) = \sum_{n=1}^{\infty}(b_n \sin n\omega x)$
where $b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin n\omega x\,dx$

Hint: $\omega = \frac{2\pi}{p} = \frac{\pi}{L}$

10.9 LINEARITY PROPERTY

Linearity property is also known as sum and scalar multiple property. It can be applied to simplify the calculation of Fourier series in some cases. For example,

(i) Fourier series of a function #1, $g(x)$ is given:

$g(x) = a_{0,g} + \sum_{n=1}^{\infty}\left(a_{n,g} \cos n\omega x + b_{n,g} \sin n\omega x\right)$

(ii) Fourier series of a function #2, $h(x)$ is given:

$h(x) = a_{0,h} + \sum_{n=1}^{\infty}\left(a_{n,h} \cos n\omega x + b_{n,h} \sin n\omega x\right)$

(iii) If a function #3, $f(x)$ is comprised of $g(x)$ & $h(x)$ through linear superposition:

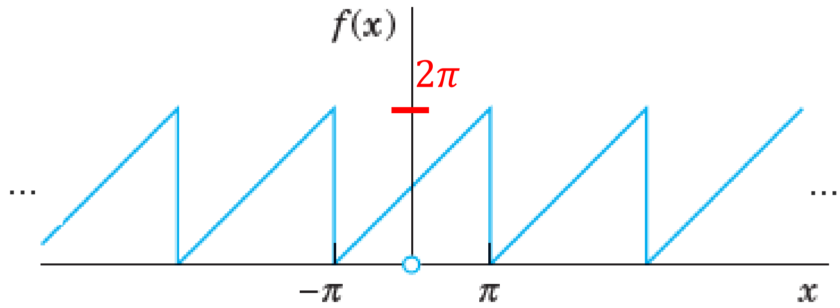
$$f(x) = mg(x) + nh(x)$$

Then, its Fourier series coefficients can be obtained by linearity property:

$$f(x) = a_{0,f} + \sum_{n=1}^{\infty}\left(a_{n,f} \cos n\omega x + b_{n,f} \sin n\omega x\right)$$

where $a_{0,f} = ma_{0,g} + na_{0,h}$
$a_{n,f} = ma_{n,g} + na_{n,h}$
$b_{n,f} = mb_{n,g} + nb_{n,h}$

Example: Find the Fourier series of the sawtooth wave



$$f(x) = \{x + \pi \quad \text{if} \quad -\pi < x < \pi$$

and

$$f(x) = f(x + n(2\pi)) \text{ where } n = 1,2,\ldots,\infty$$

Important parameters: $p = 2\pi, L = \pi, \omega = \frac{2\pi}{2\pi} = 1, f = \frac{1}{2\pi}$

Solution #1: Conventional Fourier Series Approach

$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}(x + \pi\,)dx = \pi$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, cos\, nx\;\; dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + \pi\,)\, cos\, nx\;\; dx = 0$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, sin\, nx\;\; dx = -\frac{2}{n} cosn\pi = \begin{cases} \frac{2}{n} & for\ odd\ n \\ -\frac{2}{n} & for\ even\ n \end{cases}$

Comment: Time consuming if integration of $f(x)$ is difficult.

Solution #2: Solving Fourier Series using Linearity Property

Observation: $f(x) = x + \pi$ is a *linear superposition between function, x & function π* with constant $m, n = 1$

Hence, we can use linearity property to simplify the calculation.

(i) ***Let*** $g(x) = x$

Since $g(-x) = -g(x)$, *thus it is an* <u>*Odd Function*</u> *and we can* reduced Fourier series into <u>Fourier Sine series</u>.

$$g(\boldsymbol{x}) = \text{Fourier Sine series} = \sum_{n=1}^{\infty} (b_{n,g} \sin nx)$$

$$b_{n,g} = \frac{1}{L}\int_{-L}^{L} g(x) \sin nx \; dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x . \sin nx \; dx$$

Integration by part:

$$Let \;\; u = x; dv = \sin(nx)\, dx$$

$$b_{n,g} = \left[x\left(\frac{-\cos(n\pi)}{n}\right)\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\left(\frac{-\cos(n\pi)}{n}\right)dx$$

$$= -\frac{2}{n} cosn\pi$$

(ii) ***Let*** $h(\boldsymbol{x}) = \pi$

Since $h(-\pi) = h(\pi)$, *thus it is an* <u>*Even Function*</u> *and we can* reduced Fourier series into <u>Fourier Cosine series</u>.

$$h(\boldsymbol{x}) = \text{Fourier Cosine series} = a_{0,h} + \sum_{n=1}^{\infty} (a_{n,h} \cos nx)$$

$$a_{0,h} = \frac{1}{2L}\int_{-L}^{L} h(x)\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \pi \, dx = \frac{1}{2\pi}[\pi x]_{-\pi}^{\pi} = \pi$$

$$a_{n,h} = \frac{1}{L}\int_{-L}^{L} h(x) \cos nx \; dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \pi \cos nx \; dx = \frac{1}{\pi}\left[\frac{\pi sinnx}{n}\right]_{-\pi}^{\pi} = \frac{2\sin(n\pi)}{n} = 0$$

(iii) Since $f(x) = superposition\ of\ g(\boldsymbol{x})\ \&\ h(\boldsymbol{x}) = g(\boldsymbol{x}) + h(\boldsymbol{x})$,
The Fourier coefficients of $f(x)$ can be obtained from linearity property.

$$f(x) = a_0 + \textstyle\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = a_{0,h} + a_{0,g} = \pi + 0 = \pi$$

$$a_n = a_{n,h} + a_{n,g} = 0 + 0 = 0$$

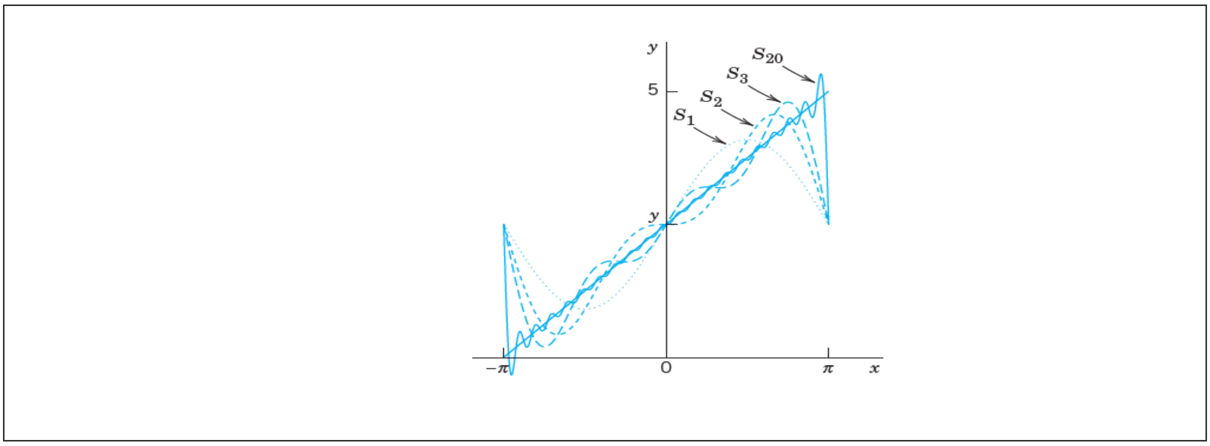
$$b_n = b_{n,h} + b_{n,g} = 0 + \left(-\frac{2}{n} cosn\pi\right) = -\frac{2}{n} cosn\pi$$

Final Solution:

$$f(x) = \pi + \textstyle\sum_{n=1}^{\infty}\left(-\frac{2}{n} cosn\pi \sin nx\right)$$

$$where \cos n\pi = \begin{cases} -1 & \text{for odd } n \\ 1 & \text{for even } n, \end{cases}$$

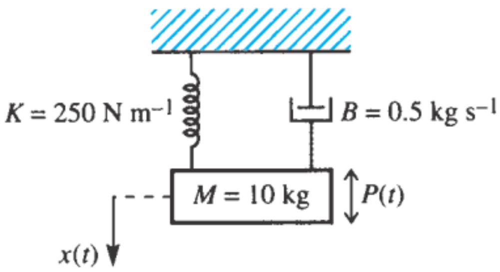
$$\therefore f(x) = \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right)$$

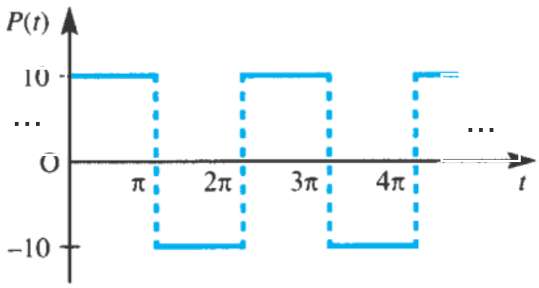


10.10 APPLICATION OF FOURIER SERIES #4: TO SOLVE NON-HOMOGENEOUS ODE WITH PERIODIC EXCITATION

Previously, we demonstrated the way to solve nonhomogeneous ODE with various types of excitations such as impulse function, unit step function, trigonometric function, exponential function, polynomial function and etc. In this section, we will demonstrate how to solve the nonhomogeneous ODE with periodic function.

For example, find the response solution of the mechanical system due to the periodic rectangular wave excitation below.

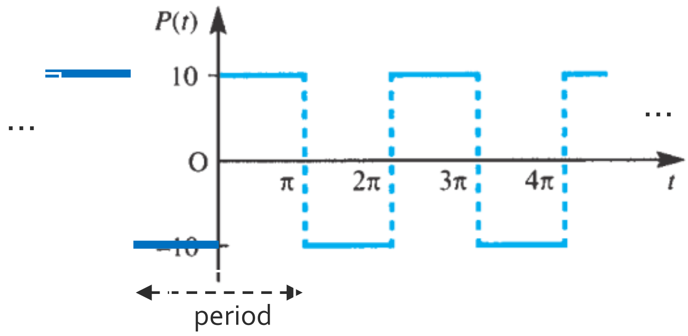




$$10\frac{d^2x}{dt^2} + 0.5\frac{dx}{dt} + 250x = P(t)$$

where $P(t) = \begin{cases} 10 & 0 \le t \le \pi \\ -10 & \pi \le t \le 2\pi \end{cases}$ & $P(t) = P(t + 2\pi n)$. The initial conditions are zero.

Step 1: Retrieve all important parameter from the periodic function



$p = 2\pi,\ L = \pi,\ \omega = \frac{2\pi}{2\pi} = 1,\ f = \frac{1}{2\pi}$

Step 2: Find the Fourier series expression of the periodic function.

Observation: Since the right side of the figure is upside down to the left side of the figure, it is an odd function. Therefore, the Fourier series expression can be simplified as the Fourier Sine series.

Fourier Sine series

$$\textit{Fourier series}, P(t) = \textit{Fourier Sine series}, P_s(t)$$
$$= \sum_{n=1}^{\infty}(b_n \sin n\omega t)$$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\, sin\, n\omega x\ dx = \frac{1}{L}\int_{0}^{2L} f(x)\, sin\, n\omega x\ dx$,where $n = 1,2,3, \ldots$

$= \frac{1}{\pi}\left[\int_0^{\pi} 10\, sin\, nx\ dx + \int_{\pi}^{2\pi} -10\, sin\, nx\ dx\right]$,where $n = 1,2,3, \ldots$

$= \frac{10}{\pi}\left[\frac{-cosnx}{n}\right]_0^{\pi} + \frac{10}{\pi}\left[\frac{+cosnx}{n}\right]_{\pi}^{2\pi}$

$= \frac{10}{n\pi}\left(-\cos n\pi - (-cos\, 0)\right) + \frac{10}{n\pi}\left(\cos(2n\pi) - (cos\, n\pi)\right)$

$= \frac{10}{n\pi}\left(2 - 2cos(n\pi)\right)$

$= \frac{20}{n\pi}\left(1 - cos(n\pi)\right)$

$b_n = \begin{cases} 40/n\pi & if\ n = \text{odd number} \\ 0 & if n = \text{even number} \end{cases}$

$\therefore\ P(t) = \frac{40}{\pi}\sin t + \frac{40}{3\pi}\sin 3t + \frac{40}{5\pi}\sin 5t + \frac{40}{7\pi}\sin 7t + \cdots =$

where $f(t)$ is valid for any interval $-\infty \le t \le \infty$

Step 3: Linear Superposition Concept

$$P(t) = \frac{40}{\pi}\sin t + \frac{40}{3\pi}\sin 3t + \frac{40}{5\pi}\sin 5t + \frac{40}{7\pi}\sin 7t + \cdots$$

(Multiple sinusoidal forces acting simultaneously)

System modeling

$$m\ddot{y} + c\dot{y} + ky = P_1(t) + P_2(t) + \cdots + P_\infty(t)$$

(Total responses can be obtained by linear superposition)

We can simplify the input force as $P(t) = \sum_{n=1}^{\infty} \frac{40}{(2n-1)\pi} \sin((2n-1)\,t)$

As a rule of thumb, $P(t) \approx P_1(t) + P_2(t) + \cdots + P_{20}(t)$

Step 4: Solve the ODE using method of Undetermined Coefficient (*Recall Math 1*)

$$10\frac{d^2x}{dt^2} + 0.5\frac{dx}{dt} + 250x = P(t)$$

where $P(t)$ represents the external forcing function, which is a periodic function.

The total solution, $x_{total} = x_{complementary} + x_{particular}$

where the complementary solution, x_c can be obtained from the homogeneous part of the ODE while the particular solution can be obtained from the non-homogeneous part of the ODE.

Note: Complementary solution is also known as transient solution while the particular solution is also known as steady state solution.

***Step 4.1**: Solving the homogeneous part of the ODE*

$$10\frac{d^2x}{dt^2} + 0.5\frac{dx}{dt} + 250x = 0$$

Assume the complementary solution is in the form of $x_c = e^{mt}$

Then, we obtain the characteristic equation, $10m^2 + 0.5m + 250 = 0$

$$m = -\frac{1}{40} \pm i\frac{\sqrt{39999}}{40}$$

The complementary solution, $x_c = c_1 e^{\left(-\frac{1}{40}+i\frac{\sqrt{39999}}{40}\right)t} + c_2 e^{\left(-\frac{1}{40}-i\frac{\sqrt{39999}}{40}\right)t}$

Or it can be represented in the trigonometric format $x_c = e^{-\frac{1}{40}t}\left(c_1 cos\frac{\sqrt{39999}}{40}t + c_2 sin\frac{\sqrt{39999}}{40}t\right)$

Given that the initial condition is zero, $x_c(0) = 0$ & $\dot{x}_c(0) = 0$

$x_c(0) = e^{-\frac{1}{40}(0)}\left(c_1 cos\frac{\sqrt{39999}}{40}(0) + c_2 sin\frac{\sqrt{39999}}{40}(0)\right) = c_1 = 0$

Thus, $x_c = e^{-\frac{1}{40}t}\left(c_2 sin\frac{\sqrt{39999}}{40}t\right)$

$\dot{x}_c = -\frac{1}{40}e^{-\frac{1}{40}t}\left(c_2 sin\frac{\sqrt{39999}}{40}t\right) + e^{-\frac{1}{40}t}\left(c_2\frac{\sqrt{39999}}{40}cos\frac{\sqrt{39999}}{40}t\right)$

$\dot{x}_c(0) = -\frac{1}{40}e^{-\frac{1}{40}(0)}\left(c_2 sin\frac{\sqrt{39999}}{40}(0)\right) + e^{-\frac{1}{40}(0)}\left(c_2\frac{\sqrt{39999}}{40}cos\frac{\sqrt{39999}}{40}(0)\right) = c_2\frac{\sqrt{39999}}{40} = 0$

$c_2 = 0$

For zero initial condition, the complementary solution, $x_c = 0$

***Step 4.2**: Solving the non-homogeneous part of the ODE*

$$10\frac{d^2x}{dt^2} + 0.5\frac{dx}{dt} + 250x = \sum_{n=1}^{\infty}\frac{40}{(2n-1)\pi}sin((2n-1)\,t)$$

Based on the RHS function (i.e. a periodic function), the possible particular solution is proposed in the form of Fourier series:

RHS function	Possible Particular Solution y_p
(i) Periodic Function, e.g. A rectangular wave function $P(t) = \begin{cases} 10 & 0 \le t \le \pi \\ -10 & \pi \le t \le 2\pi \end{cases}$	$x_p = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$ where a_0, a_n & b_n are the three unknowns (also known as undetermined coefficient) to be solved.
Example:	$x_p = a_0 + \sum_{n=1}^{\infty}(a_n \cos((2n-1)t) + b_n \sin((2n-1)t))$

$P(t) = \sum_{n=1}^{\infty} \frac{40}{(2n-1)\pi} \sin((2n-1)\,t)$	**Comment**: No treatment is needed after we compare the coefficients obtained from the homogeneous and non-homogeneous parts of ODE. $(2n-1) \neq -\frac{1}{40} \pm i\frac{\sqrt{39999}}{40}$

- Differentiate the particular solution,

$$x'_p = 0 + \sum_{n=1}^{\infty} \left(-(2n-1)a_n \sin((2n-1)t) + (2n-1)b_n \cos((2n-1)t)\right)$$

$$x''_p = \sum_{n=1}^{\infty} \left(-(2n-1)^2 a_n \cos\big((2n-1)t\big) - (2n-1)^2 b_n \sin\big((2n-1)t\big)\right)$$

- Substitute to the ODE

$$10\frac{d^2x}{dt^2} + 0.5\frac{dx}{dt} + 250x = \sum_{n=1}^{\infty} \frac{40}{(2n-1)\pi} \sin((2n-1)\,t)$$

$$10\left[\sum_{n=1}^{\infty} \left(-(2n-1)^2 a_n \cos\big((2n-1)t\big) - (2n-1)^2 b_n \sin\big((2n-1)t\big)\right)\right]$$

$$+0.5\left[\sum_{n=1}^{\infty} \left(-(2n-1)a_n \sin\big((2n-1)t\big) + (2n-1)b_n \cos\big((2n-1)t\big)\right)\right]$$

$$+250\left[a_0 + \sum_{n=1}^{\infty} \left(a_n \cos((2n-1)t) + b_n \sin((2n-1)t)\right)\right] = \sum_{n=1}^{\infty} \frac{40}{(2n-1)\pi} \sin((2n-1)\,t)$$

- Rearrange,

$$\sum_{n=1}^{\infty} \Big((-10)(2n-1)^2 a_n \cos\big((2n-1)t\big) + (0.5)(2n-1)b_n \cos\big((2n-1)t\big) + 250a_n \cos((2n-1)t)\Big)$$

$$+\sum_{n=1}^{\infty} \Big((-10)(2n-1)^2 b_n \sin\big((2n-1)t\big) - (0.5)(2n-1)a_n \sin\big((2n-1)t\big) + 250b_n \sin((2n-1)t)\Big)$$

$$+250a_0 = \sum_{n=1}^{\infty} \frac{40}{(2n-1)\pi} \sin((2n-1)\,t)$$

- Compare coefficient of t^0:

$$250a_0 = 0$$

Thus, $a_0 = 0$

- Compare coefficient of $\cos\big((2n-1)t\big)$:

$$(-10)(2n-1)^2 a_n + (0.5)(2n-1)b_n + 250a_n = 0$$

$$b_n = \frac{(10)(2n-1)^2 - 250}{(0.5)(2n-1)} a_n$$

where $n = 1,2,3, \dots$

- Compare coefficient of $\sin((2n-1)t)$:

$$(-10)(2n-1)^2 b_n - (0.5)(2n-1)a_n + 250b_n = \frac{40}{(2n-1)\pi}$$

$$[(-10)(2n-1)^2 + 250]b_n - (0.5)(2n-1)a_n = \frac{40}{(2n-1)\pi}$$

$$[(-10)(2n-1)^2 + 250]\left[\frac{(10)(2n-1)^2 - 250}{(0.5)(2n-1)} a_n\right] - (0.5)(2n-1)a_n = \frac{40}{(2n-1)\pi}$$

$$[(-10)(2n-1)^2 + 250][(10)(2n-1)^2 - 250]a_n - (0.5)^2(2n-1)^2 a_n = \frac{20}{\pi}$$

$$a_n = \frac{20}{\pi([(-10)(2n-1)^2 + 250][(10)(2n-1)^2 - 250] - (0.5)^2(2n-1)^2)}$$

$$a_n = \frac{20}{\pi([(-100)(2n-1)^4 + (5000)(2n-1)^2 - 62500] - (0.5)^2(2n-1)^2)}$$

$$a_n = \frac{20}{\pi((-100)(2n-1)^4 + (4999.75)(2n-1)^2 - 62500)}$$

where $n = 1,2,3, \dots$

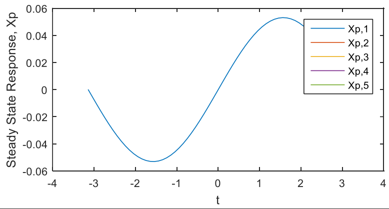
$$b_n = \frac{(10)(2n-1)^2 - 250}{(0.5)(2n-1)} \frac{20}{\pi((-100)(2n-1)^4 + (4999.75)(2n-1)^2 - 62500)}$$

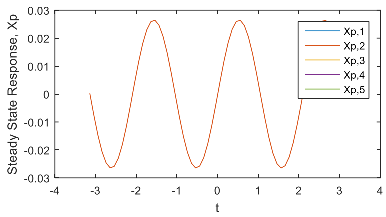
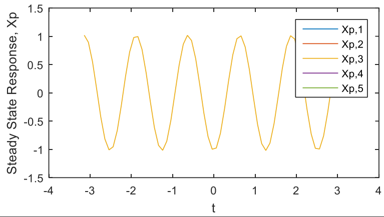
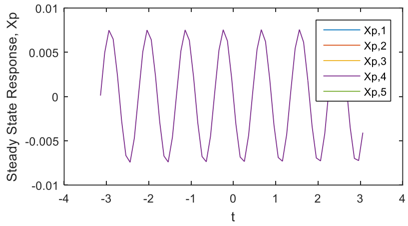
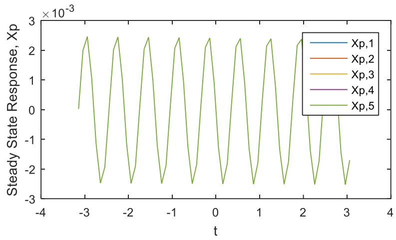
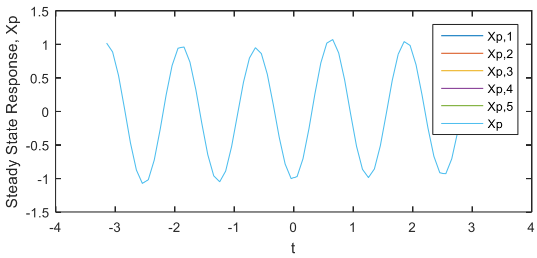
- Thus, the particular solution

$$x_p = a_0 + \sum_{n=1}^{\infty} (a_n \cos(2n-1)t + b_n \sin(2n-1)t)$$

$$= \sum_{n=1}^{\infty} \left(\begin{array}{l} \dfrac{20}{\pi((-100)(2n-1)^4 + (4999.75)(2n-1)^2 - 62500)} \cos(2n-1)t \\ + \dfrac{(10)(2n-1)^2 - 250}{(0.5)(2n-1)} \dfrac{20}{\pi((-100)(2n-1)^4 + (4999.75)(2n-1)^2 - 62500)} \sin(2n-1)t \end{array} \right)$$

An approximation of the solution is given by linear superposition of the first 5 terms response solution.

Term, n	Particular solution due to nth forcing function	Observation
1	$x_{p,1} = -1.10524x10^{-4} cost + 0.0531 sint$ 	Magnitude, $\lvert x_{p,1} \rvert = \sqrt{(-1.10524x10^{-4})^2 + (0.0531)^2} = 0.053$
2	$x_{p,2} = -2.4866x10^{-4} cos3t + 0.0265 sin3t$	Magnitude, $\lvert x_{p,2} \rvert = \sqrt{(-2.4866x10^{-4})^2 + (0.0265)^2}$

		$= 0.027$
3	$x_{p,3} = -1.0190cos5t + 0sin5t$ 	Magnitude, $\lvert x_{p,3}\rvert$ $= \sqrt{(-1.0190)^2 + (0)^2}$ $= 1.0190$
4	$x_{p,4} = -1.1050x10^{-4}cos7t - 7.577x10^{-3}sin7t$ 	Magnitude, $\lvert x_{p,4}\rvert$ $= \sqrt{\begin{matrix}(-1.1050x10^{-4})^2\\+(-7.577x10^{-3})^2\end{matrix}}$ $= 0.0076$
5	$x_{p,5} = -2.0300x10^{-5}cos9t - 2.526x10^{-3}sin9t$ 	Magnitude, $\lvert x_{p,4}\rvert$ $= \sqrt{\begin{matrix}(-2.0300x10^{-5})^2\\+(-2.526x10^{-3})^2\end{matrix}}$ $= 0.0025$
⋮	⋮	⋮
∞	$x_p = x_{p,1} + x_{p,2} + x_{p,3} + x_{p,4} + x_{p,5} + \cdots x_{p,\infty}$ 	Magnitude, $\lvert x_p\rvert$ $\lvert x_p\rvert = \lvert x_{p,1}\rvert + \lvert x_{p,2}\rvert + \lvert x_{p,3}\rvert + \cdots$ $\lvert x_p\rvert \approx \lvert x_{p,3}\rvert$ **Note** that the total particular solution is dominated by the 3th term response. This is because the 3th forcing function has exciting frequency $(\omega = 5)$ equal to the natural frequency $(\sqrt{\frac{k}{m}} = \sqrt{\frac{250}{10}} = 5)$. Resonance occurs!

Step 4.3: $x_{total} = x_{complementary} + x_{particular}$

$$
\begin{aligned}
x_{total} = 0 &+ (-1.10524x10^{-4}cost + 0.0531sint) + (-2.4866x10^{-4}cos3t + 0.0265sin3t) \\
&+ (-1.0190cos5t + 0sin5t) + (-1.1050x10^{-4}cos7t - 7.577x10^{-3}sin7t) \\
&+ (-2.0300x10^{-5}cos9t - 2.526x10^{-3}sin9t) +
\end{aligned}
$$

$$
\left(\begin{array}{c}
\frac{20}{\pi\left((-100)(2n-1)^4 + (4999.75)(2n-1)^2 - 62500\right)}\cos(2n-1)t \\
+\frac{(10)(2n-1)^2 - 250}{(0.5)(2n-1)}\frac{20}{\pi\left((-100)(2n-1)^4 + (4999.75)(2n-1)^2 - 62500\right)}\sin(2n-1)t
\end{array}\right) + \cdots
$$

where $n = 6,7,8, \ldots$

Exercise:

Repeat the example by letting the damping coefficient be zero. Observe the severity of the vibration level after the changes of parameters.

$$
10\frac{d^2x}{dt^2} + 250x = P(t)
$$

where $P(t) = \begin{cases} 10 & 0 \le t \le \pi \\ -10 & \pi \le t \le 2\pi \end{cases}$ & $P(t) = P(t + 2\pi n)$. The initial conditions are zero.