SOLVING PARTICULAR SOLUTION OF LAPLACE EQUATION

WEEK 13: SOLVING PARTICULAR SOLUTION OF LAPLACE EQUATION

13.1 EIGENVALUE AND EIGENFUNCTION OF ODE (BACKGROUND - EXTRA INFO)

Find all the eigenvalues and eigenfunction of the following ODE, where $Y'' = \frac{d^2Y}{dy^2}$

$Y'' + \lambda Y = 0$ where $Y(0) = 0$ and $Y(2\pi) = 0$

The ODE above can be transformed to an eigenvalue problem:
Let $Y = A_1 sin(\omega t + \theta_1)$ and

$$Y'' = -\omega^2 A_1 sin(\omega t + \theta_1) = -\omega^2 Y$$

$-\omega^2 Y + \lambda Y = 0$
$(\lambda - \omega^2)Y = 0$

The solution Y can't be zero and hence $|\lambda - \omega^2| = 0$,
where the **eigenvalue, $\boldsymbol{\lambda}$** $= \omega^2$ &
the **corresponding solution $\boldsymbol{Y}$ is the eigenfunction of the ODE**.

Recall that eigenfunction represents each of a set of independent functions, which are the solutions to a given differential equation.

Case	General solution of the ODE	Particular solution of the ODE
Case #1: (λ=0)	$Y'' = 0$ Let r =root Characteristic equation: $r^2 = 0$ Repeated roots: $r_1 = 0,\ r_2 = 0$ $Y(y) = c_1 e^{0y} + c_2 y e^{0y}$ $\therefore Y(y) = c_1 + c_2 y$	Using boundary condition, $Y(0) = c_1 + c_2(0) = 0$ $c_1 = 0$ $\rightarrow Y = c_2 y$ $Y(2\pi) = 0 = c_2$ $\therefore Y = 0$ (No solution if λ=0)
Case #2: ($\lambda = -\alpha^2$) $\alpha > 0$	$Y'' + (-\alpha^2)Y = 0$ Let r =root Characteristic equation: $r^2 - \alpha^2 = 0$ $r = \pm\sqrt{\alpha^2} = \pm\alpha$ Distinct roots: $r_1 = \alpha,\ r_2 = -\alpha$ $\therefore Y = c_3 cosh(\alpha y) + c_4 sinh(\alpha y)$ *Hint: Refer section 12.4*	Using boundary condition, $Y(0) = c_3 \cosh(0) + c_4 \sinh(0) = 0$ $c_3(1) + c_4(0) = 0$ $c_3 = 0$ $\rightarrow Y = c_4 sinh\alpha y$ $Y(2\pi) = c_4 sinh\alpha(2\pi) = 0$ Since $\alpha(2\pi) > 0$ & $sinh(+ve)$ is never equal to zero for all $\alpha(2\pi)$, thus $c_4 = 0$

		$\therefore Y = 0$ (No solution if $\lambda = -\alpha^2$)
Case #3: $(\lambda = +\alpha^2)$ $\alpha > 0$	$$Y'' + (\alpha^2)Y = 0$$ Let r =root Characteristic equation: $$r^2 + \alpha^2 = 0$$ $$r = \pm\sqrt{-\alpha^2} = \pm\alpha i$$ Complex conjugate roots: $$r_1 = \alpha i, \quad r_2 = -\alpha i$$ $\therefore Y(y) = c_5\cos(\alpha y) + c_6\sin(\alpha y)$ *Hint: Refer section 12.4*	Using boundary condition, $$Y(0) = c_5\cos(0) + c_6\sin(0) = 0$$ $$c_5(1) + c_6(0) = 0$$ $$c_5 = 0$$ $\rightarrow Y = c_6\sin(\alpha y)$ $$Y(2\pi) = c_6 sin(2\pi\alpha) = 0$$ **Possibility 1:** If $c_6 = 0$, we will get no solution, $Y = 0$. **Possibility 2:** So, we check if $sin(2\pi\alpha)$ can be zero. Since $2\pi\alpha > 0$ & $sin(2\pi\alpha) = 0$ when $2\pi\alpha = n\pi$, where integer $n = 1,2,3,\ldots$ Then, $c_6 \neq 0$ in this condition. $\therefore Y_n = c_{6,n}\sin(\alpha y)$ where the $\alpha = \frac{n}{2}$ for $n = 1,2,3,\ldots$ (We have solution if $\lambda = +\alpha^2$) Think: Can the solution valid for $n = \cdots, -2, -1, 0$? Hint: $2\pi\alpha > 0$

Eigenvalue for the ODE, $\lambda = +\alpha^2 = \frac{n^2}{4}$ for $n = 1,2,3,\ldots$

Eigenfunction for the ODE, $Y_n = c_{6,n}\sin(\frac{n}{2}y)$ for $n = 1,2,3,\ldots$

$n = 1 \rightarrow Y_1 = c_{6,1}\sin(\frac{1}{2}y)$

$n = 2 \rightarrow Y_2 = c_{6,2}\sin(\frac{2}{2}y)$

Thus, we have infinite solutions in the 3rd case, by using the superposition principle:

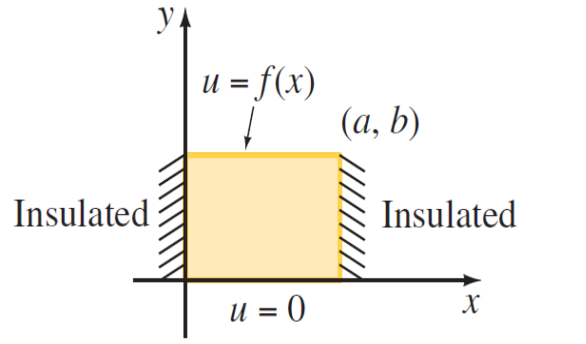
$Y_{total} = Y_1 + Y_2 + \cdots = \sum_{n=1}^{\infty} c_{6,n}\sin(\frac{n}{2}y)$

Note that $c_{6,n}$ can be solved further with Fourier series expansion & additional initial/boundary condition. Then, the complete particular solution can be obtained.

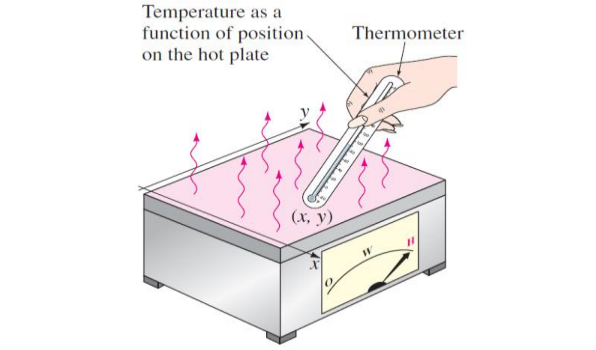
The similar concepts discussed in section 13.1 can be used to solve the PDE problem.

13.2 SOLVING PARTICULAR SOLUTION OF ELLIPTIC PDE (LAPLACE EQUATION)

Consider a hot place of area (xy), find the steady state temperature distribution over the x and y location, i.e. $u(x, y)$.



Top view of the 2D heated plate



3D view of the heated plate

- **Governing equation for the 2D Laplace equation**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

- **Boundary condition 1 & 2:** $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0\,, \left.\frac{\partial u}{\partial x}\right|_{x=a} = 0$ for $0 < y < b$

- **Boundary condition 3 & 4:** $u(x,\ 0) = 0\,, u(x,\ b) = f(x)$ for $0 < x < a$

Solution:

Step 1: *Using separation of variable method:* Let $u(x, y) = X(x)Y(y)$

$$X''Y + XY'' = 0$$

Step 2: *Obtain 2 ODE equations*

$$\frac{Y''}{-Y} = \frac{X''}{X} = -\lambda$$

$$Y'' - \lambda Y = 0 \text{ --- (ODE \#1)}$$

$$X'' + \lambda X = 0 \text{ --- (ODE \#2)}$$

Case	ODE #1	ODE #2	$u(x, y) = X(x)Y(y)$
Case #1: (λ=0)	$Y'' = 0$ Let r =root Characteristic equation: $r^2 = 0$ Repeated roots: $r_1 = 0,\quad r_2 = 0$ $Y(y) = c_1 e^{0y} + c_2 y e^{0y}$	$X'' = 0$ Let r =root Characteristic equation: $r^2 = 0$ Repeated root: $r_1 = 0$, $r_2 = 0$ $X(x) = c_3 e^{0x} + c_4 x e^{0x}$	$\therefore u_1 = X_1(x)Y_1(y)$ $= (c_1 + c_2 y)(c_3 + c_4 x)$

	$\therefore Y(y) = c_1 + c_2 y$	$\therefore X(x) = c_3 + c_4 x$	
Case #2: (λ= $-\alpha^2$) $\alpha > 0$	$Y'' + (\alpha^2)Y = 0$ Let r =root Characteristic equation: $r^2 + \alpha^2 = 0$ $r = \pm\sqrt{-\alpha^2} = \pm\alpha i$ Complex conjugate roots: $r_1 = \alpha i,\ r_2 = -\alpha i$ $\therefore Y(y) = c_5 cos(\alpha y) + c_6 sin(\alpha y)$	$X'' - \alpha^2 X = 0$ Let r =root Characteristic equation: $r^2 - \alpha^2 = 0$ $r = \pm\sqrt{\alpha^2} = \pm\alpha$ Distinct roots: $r_1 = \alpha, \quad r_2 = -\alpha$ $\therefore X(x) = c_7 cosh(\alpha x) + c_8 sinh(\alpha x)$	$\therefore u_2 = X_2(x)Y_2(y)$ $= \left(c_5 cos(\alpha y) + c_6 sin(\alpha y)\right)(c_7 cosh(\alpha x) + c_8 sinh(\alpha x))$
Case #3: (λ= $+\alpha^2$) $\alpha > 0$	$Y'' + (-\alpha^2)Y = 0$ Let r =root Characteristic equation: $r^2 - \alpha^2 = 0$ $r = \pm\sqrt{\alpha^2} = \pm\alpha$ Distinct roots: $r_1 = \alpha, \quad r_2 = -\alpha$ $\therefore Y(y) = c_9 cosh(\alpha y) + c_{10} sinh(\alpha y)$	$X'' + (\alpha^2)X = 0$ Let r =root Characteristic equation: $r^2 + \alpha^2 = 0$ $r = \pm\sqrt{-\alpha^2} = \pm\alpha i$ Complex conjugate roots: $r_1 = \alpha i,\ r_2 = -\alpha i$ $\therefore X(x) = c_{11} cos(\alpha x) + c_{12} sin(\alpha x)$	$\therefore u_3 = X_3(x)Y_3(y)$ $= \left(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y)\right)\left(c_{11} cos(\alpha x) + c_{12} sin(\alpha x)\right)$

In fact, we can find the general PDE solution to the problem by using superposition principle:

$$u(x,y) = \underbrace{(c_1 + c_2 y)(c_3 + c_4 x)}_{Solution\ of\ Case\ 1} + \underbrace{\left(c_5 cos(\alpha y) + c_6 sin(\alpha y)\right)(c_7 cosh(\alpha x) + c_8 sinh(\alpha x))}_{Solution\ of\ Case\ 2}$$

$$+ \underbrace{\left(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y)\right)\left(c_{11} cos(\alpha x) + c_{12} sin(\alpha x)\right)}_{Solution\ of\ Case\ 3}$$

where there are 12 unknown coefficients ($c_1 - c_{12}$). Next we will continue to solve those unknowns by applying the initial/ boundary conditions.

To apply the following boundary conditions, differentiation of the PDE solution is needed.

Boundary condition (BC) #1: $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0$, BC #2: $\left.\frac{\partial u}{\partial x}\right|_{x=a} = 0$

Case	Differentiation of $u(x,y) = X(x)Y(y)$ wrt x
Case #1: (λ=0)	$u_1 = X_1(x)Y_1(y)$ $= (c_1 + c_2 y)(c_3 + c_4 x)$ $\frac{\partial u_1}{\partial x} = (c_1 + c_2 y)(c_4)$ Applying BC #1 or BC #2, we get $(c_1 + c_2 y)(c_4) = 0$

	Since $(c_1 + c_2 y) \neq 0, c_4 = 0$ $\therefore u_1(x,y) = (c_1 + c_2 y)(c_3) = (A_1 + B_1 y)$
Case #2: ($\lambda = -\alpha^2$) $\alpha > 0$	$u_2 = X_2(x)Y_2(y)$ $= (c_5 cos(\alpha y) + c_6 sin(\alpha y))(c_7 cosh(\alpha x) + c_8 sinh(\alpha x))$ $\frac{\partial u_2}{\partial x} = (c_5 cos(\alpha y) + c_6 sin(\alpha y))(c_7 \alpha sinh(\alpha x) + c_8 \alpha cosh(\alpha x))$ Applying BC #1: $(c_5 cos(\alpha y) + c_6 sin(\alpha y))(c_8 \alpha) = 0$ Since $(c_5 cos(\alpha y) + c_6 sin(\alpha y)) \neq 0, \alpha \neq 0$, thus $c_8 = 0$ $\rightarrow u_2(x,y) = (c_5 cos(\alpha y) + c_6 sin(\alpha y))(c_7 cosh(\alpha x))$ $\rightarrow \frac{\partial u_2}{\partial x} = (c_5 cos(\alpha y) + c_6 sin(\alpha y))(c_7 \alpha sinh(\alpha x))$ Applying BC #2: $(c_5 cos(\alpha y) + c_6 sin(\alpha y))(c_7 \alpha sinh(\alpha a)) = 0$ Since $(c_5 cos(\alpha y) + c_6 sin(\alpha y)) \neq 0, \alpha \neq 0, sinh(\alpha a) \neq 0$ for $\alpha a > 0$ Hence, $c_7 = 0$ $\therefore u_2(x,y) = 0$ (No solution)
Case #3: ($\lambda = +\alpha^2$) $\alpha > 0$	$\therefore u_3 = X_3(x)Y_3(y)$ $= (c_9 cosh(\alpha y) + c_{10} sinh(\alpha y))(c_{11} cos(\alpha x) + c_{12} sin(\alpha x))$ $\frac{\partial u_3}{\partial x} = (c_9 cosh(\alpha y) + c_{10} sinh(\alpha y))(-c_{11} \alpha sin(\alpha x) + c_{12} \alpha cos(\alpha x))$ Applying BC #1: $(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y))(c_{12} \alpha) = 0$ Since $(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y)) \neq 0$ & $\alpha \neq 0$ Hence, $c_{12} = 0$ $\rightarrow u_3 = (c_9 cosh(\alpha y) + c_{10} sinh(\alpha y))(c_{11} cos(\alpha x))$ $\rightarrow \frac{\partial u_3}{\partial x} = (c_9 cosh(\alpha y) + c_{10} sinh(\alpha y))(-c_{11} \alpha sin(\alpha x))$ Applying BC #2: $(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y))(-c_{11} \alpha sin(\alpha a)) = 0$ Since $(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y)) \neq 0$ & $\alpha \neq 0$ $c_{11} \neq 0$ *when* $sin(\alpha a) = 0$ for $\alpha a = n\pi$, where $\alpha = \frac{n\pi}{a}, n = 1,2,3 \ldots$ There are infinite solutions in Case #3: $u_{3,n} = (c_{9,n} cosh(\frac{n\pi}{a} y) + c_{10,n} sinh(\frac{n\pi}{a} y))(c_{11,n} cos(\frac{n\pi}{a} x))$ $u_{3,n} = (A_{3,n} cosh(\frac{n\pi}{a} y) + B_{3,n} sinh(\frac{n\pi}{a} y))(cos(\frac{n\pi}{a} x))$ *where* $n = 1,2,3,\ldots$

In summary, the eigenvalue and eigenfunction of the PDE for each case are listed below:

Case	PDE solution	Eigenvalue and eigenfunction of PDE
Case #1:	$u_1(x,y) = A_1 + B_1 y$	Eigenvalue, λ=0

(λ=0)		Eigenfunction $= A_1 + B_1 y$
Case #2: (λ$= -\alpha^2$) $\alpha > 0$	$u_2 = 0$	No solution hence no eigenvalue and no eigenfunction
Case #3: (λ$= +\alpha^2$) $\alpha > 0$	$u_{3,n} = \left(A_{3,n} cosh(\frac{n\pi}{a} y) + B_{3,n} sinh(\frac{n\pi}{a} y)\right) \left(cos(\frac{n\pi}{a} x)\right))$ *where* $n = 1,2,3, \ldots$	*Eigenvalue,* $\lambda_n = +{\alpha_n}^2 = \left(\frac{n\pi}{a}\right)^2$ *Eigenfunction* $u_{3,n} = \left(A_{3,n} cosh(\frac{n\pi}{a} y) + B_{3,n} sinh(\frac{n\pi}{a} y)\right) \left(cos(\frac{n\pi}{a} x)\right)$

Step 4: Superposition Principle to find $u_{total}(x, y) = X_1Y_1 + X_2Y_2 + X_3Y_3$

$$u_{total}(x, y) = \underbrace{(A_1 + B_1 y)}_{solution\ 1\ from\ Case\ 1} + \underbrace{\sum_{n=1}^{\infty} \left(A_{3,n} cosh(\frac{n\pi}{a} y) + B_{3,n} sinh(\frac{n\pi}{a} y)\right) \left(cos(\frac{n\pi}{a} x)\right)}_{solution\ 2\ from\ Case\ 3}$$

Expanding it, we obtain

$$u_{total}(x, y) = (A_1 + B_1 y) + \sum_{n=1}^{\infty} \left(A_{3,n} cosh(\tfrac{n\pi}{a} y) \left(cos(\tfrac{n\pi}{a} x)\right)\right) + \sum_{n=1}^{\infty} \left(B_{3,n} sinh(\tfrac{n\pi}{a} y) \left(cos(\tfrac{n\pi}{a} x)\right)\right)$$

where there are 4 remaining unknowns (i.e. $A_1, B_1, A_{3,\mathrm{n}}$ & $B_{3,\mathrm{n}}$).

Step 5: Continue to apply the remaining BC & Fourier series expansion.

BC #3: $u(x,\ 0) = 0$ for $0 < x < a$

$$u_{total}(x, 0) = A_1 + \sum_{n=1}^{\infty} \left(A_{3,n} cosh(\tfrac{n\pi}{a}(0)) + B_{3,n} sinh(\tfrac{n\pi}{a}(0))\right) \left(cos(\tfrac{n\pi}{a} x)\right) = 0$$

$$= A_1 + \sum_{n=1}^{\infty} (A_{3,n}) \left(cos(\tfrac{n\pi}{a} x)\right) = 0$$

Recall Half-range Fourier Cosine Series Expansion:

$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega x)$ *where* $a_0 = \frac{1}{L} \int_0^{\tau} f(x)\, dx;$ $a_n = \frac{2}{L} \int_0^{\tau} f(x) \cos n\omega x\ dx;$

We notice $f(x) = 0$; $A_1 = \frac{1}{L} \int_0^{\tau} f(x)\, dx = 0$; $A_{3,\mathrm{n}} = \frac{2}{L} \int_0^{\tau} f(x) \cos n\omega x\ dx = 0$

$$\rightarrow u_{total}(x, y) = (B_1 y) + \sum_{n=1}^{\infty} \left(B_{3,n} sinh(\tfrac{n\pi}{a} y) \left(cos(\tfrac{n\pi}{a} x)\right)\right)$$

Step 5: Continue to apply the remaining BC & Fourier series expansion.

BC #4: $u(x,\ b) = f(x)$ for $0 < x < a$

$$u_{total}(x,b) = B_1(b) + \sum_{n=1}^{\infty}\left(B_{3,n} sinh(\frac{n\pi}{a}b)\right)\ \left(cos(\frac{n\pi}{a}x)\right) = f(x)$$

Recall Half-range Fourier Cosine Series Expansion:

$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega x)$ *where* $a_0 = \frac{1}{L}\int_0^{\tau} f(x)\, dx;$ $a_n = \frac{2}{L}\int_0^{\tau} f(x) \cos n\omega x\ dx;$

We notice $B_1 b = \frac{1}{L}\int_0^{\tau} f(x)\, dx$; $\left(B_{3,n} sinh(\frac{n\pi}{a}b)\right) = \frac{2}{L}\int_0^{\tau} f(x) \cos n\omega x\ dx$

where angular frequency, $\omega = \frac{\pi}{a}$, period, $p = \frac{2\pi}{\omega} = 2a$

Finite interval, $\tau{=}a$, half period, $L = \frac{p}{2} = \frac{2a}{2} = a$

$B_1 b = \frac{1}{a}\int_0^a f(x)\, dx$ $B_1 = \frac{\int_0^a f(x)dx}{ab}$	$B_{3,n} sinh(\frac{n\pi}{a}b) = \frac{2}{a}\int_0^a f(x) \cos n\frac{\pi}{a}x\, dx$ $B_{3,n} = \frac{2}{a sinh(\frac{n\pi}{a}b)}\int_0^a f(x) \cos n\frac{\pi}{a}x\, dx$

Thus, we have solved all the unknowns and obtain the particular PDE solution:

$$\therefore u_{total}(x,y) = B_1 y + \sum_{n=1}^{\infty}\left(B_{3,n} sinh(\frac{n\pi}{a}y)\right)\ \left(cos(\frac{n\pi}{a}x)\right)$$

$$u_{total}(x,t) = \frac{\int_0^a f(x)dx}{ab} y + \sum_{n=1}^{\infty}\left(\frac{2}{a sinh(\frac{n\pi}{a}b)}\int_0^a f(x) \cos n\frac{\pi}{a}x\, dx\, sinh(\frac{n\pi}{a}y)\right)\ \left(cos(\frac{n\pi}{a}x)\right)$$

Example: Let the temperature at the top end, $f(x) = 100$, dimension, $a = b = 1$ for the previous problem.

$$B_1 = \frac{\int_0^a f(x)dx}{ab}$$

$$B_1 = \frac{\int_0^1 100dx}{(1)(1)} = 100$$

$$B_{3,n} = \frac{2}{a\sinh(\frac{n\pi}{a}b)} \int_0^a f(x) \cos n\frac{\pi}{a}x\,dx$$

$$= \frac{2}{(1)\sinh(\frac{n\pi}{1}1)} \int_0^a 100 \cos n\frac{\pi}{1}x\,dx$$

$$= \frac{2}{\sinh(n\pi)} \int_0^1 100 \cos n\pi x\,dx$$

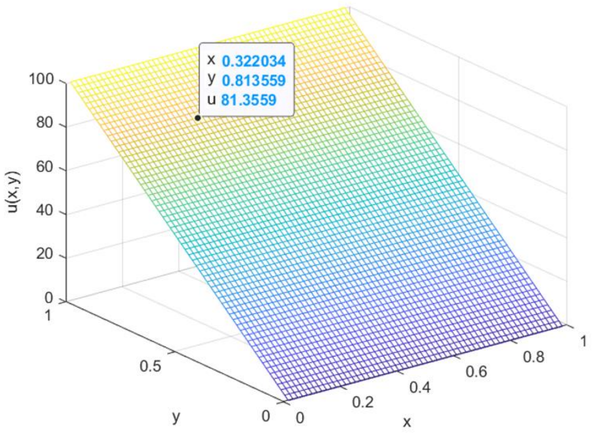
$$= \frac{2}{\sinh(n\pi)} \left[\frac{100 \sin n\pi x}{n\pi}\right]_0^1$$

$$= \frac{2}{\sinh(n\pi)} \left(\frac{100 \sin n\pi}{n\pi}\right)$$

$$\therefore u_{total}(x,y) = 100y + \sum_{n=1}^{\infty} \left(\frac{200}{\sinh(n\pi)} \left(\frac{\sin n\pi}{n\pi}\right) \sinh(n\pi y)\right) (\cos(n\pi x))$$

We can use the PDE solution to estimate the temperature distribution at any point on the heated plate.
Example: The temperature results at 60×60 points of the (x, y) locations have been plotted below:

3D plot of $u_{total}(x,y)$ wrt x- & y- axes

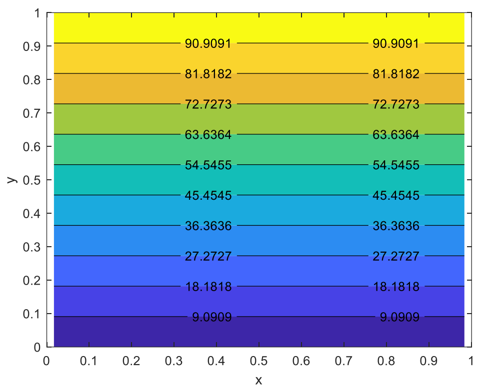


-Due to boundary conditions on 4 sides of the plates, the temperature becomes stable after certain period.

For example, $u_{total}(0.322, 0.814) \approx 81\ ^oC$. Note that 20 terms are used for plotting the graphs. For higher accuracy, more terms & more grids can be included but computational time will be increased.

Try to verify the answer:
$u_{total}(0.322, 0.814)$

2D contour plot

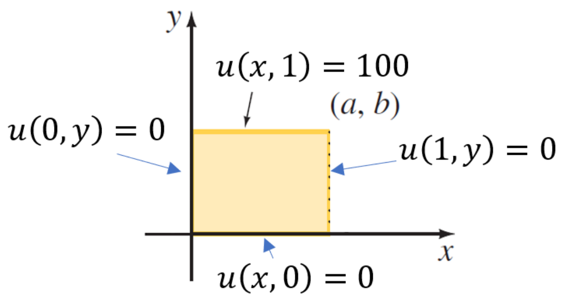
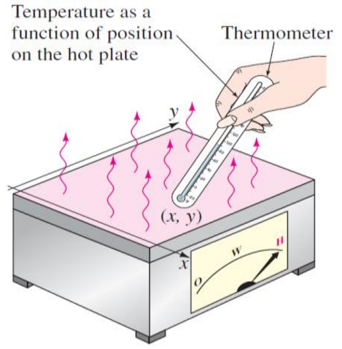


-Top view of the 3D plot with the contour, i.e. line with same magnitude.

Try to find the location of the plate that has temperature around 80 ^oC.
Hint: Orange color

$\approx \left(100(0.814) + \sum_{n=1}^{20} \left(\frac{200}{\sinh(n\pi)}\left(\frac{\sin n\pi}{n\pi}\right)\sinh(0.814n\pi)\right)\ (\cos(0.322n\pi))\right)$	

Consider a hot place of area (xy), find the <u>steady state temperature distribution over the x and y location</u>, i.e. $u(x,y)$.

 Top view of the 2D heated plate	 3D view of the heated plate

- **Governing equation for the 2D Laplace equation**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

- **Boundary condition**: $u(0,y) = 0, u(1,y) = 0$ for $0 < y < 1$

- **Boundary condition:** $u(x,\ 0) = 0\ , u(x,\ 1) = 100$ for $0 < x < 1$

Solution:

Note that the PDE equation remains the same while the boundary conditions are changing. Thus,

The general PDE solution remains the same as below:

$$u(x,y) = \underbrace{(c_1 + c_2 y)(c_3 + c_4 x)}_{Solution\ of\ Case\ 1} + \underbrace{(c_5\cos(\alpha y) + c_6\sin(\alpha y))\ (c_7\cosh(\alpha x) + c_8\sinh(\alpha x))}_{Solution\ of\ Case\ 2}$$

$$+ \underbrace{(c_9\cosh(\alpha y) + c_{10}\sinh(\alpha y))\ (c_{11}\cos(\alpha x) + c_{12}\sin(\alpha x))}_{Solution\ of\ Case\ 3}$$

where there are 12 unknown coefficients $(c_1 - c_{12})$.

To apply the following boundary conditions, differentiation of the PDE solution is no needed.

Boundary condition (BC) #1: $u(0, y) = 0$, BC #2: $u(1, y) = 0$

Case	Applying BC #1 & BC #2
Case #1: (λ=0)	$u_1 = X_1(x)Y_1(y)$ $= (c_1 + c_2 y)(c_3 + c_4 x)$ Applying BC #1, $(c_1 + c_2 y)(c_3) = 0$ Since $(c_1 + c_2 y) \neq 0$, $c_3 = 0$ $\rightarrow u_1 = (c_1 + c_2 y)(c_4 x)$ Applying BC #2, $(c_1 + c_2 y)(c_4) = 0$ Since $(c_1 + c_2 y) \neq 0$, $c_4 = 0$ $\therefore u_1(x, y) = 0$ (No Solution)
Case #2: $(\lambda = -\alpha^2)$ $\alpha > 0$	$u_2 = X_2(x)Y_2(y)$ $= (c_5 cos(\alpha y) + c_6 sin(\alpha y))(c_7 cosh(\alpha x) + c_8 sinh(\alpha x))$ Applying BC #1: $(c_5 cos(\alpha y) + c_6 sin(\alpha y))(c_7) = 0$ *Since* $(c_5 cos(\alpha y) + c_6 sin(\alpha y)) \neq 0$, *thus* $c_7 = 0$ $\rightarrow u_2(x, y) = (c_5 cos(\alpha y) + c_6 sin(\alpha y))(c_8 sinh(\alpha x))$ Applying BC #2: $(c_5 cos(\alpha y) + c_6 sin(\alpha y))(c_8 sinh(\alpha)) = 0$ *Since* $(c_5 cos(\alpha y) + c_6 sin(\alpha y)) \neq 0$, $sinh(\alpha) \neq 0$ *for* $\alpha > 0$ Hence, $c_8 = 0$ $\therefore u_2(x, y) = 0$ (No solution)
Case #3: $(\lambda = +\alpha^2)$ $\alpha > 0$	$\therefore u_3 = X_3(x)Y_3(y)$ $= (c_9 cosh(\alpha y) + c_{10} sinh(\alpha y))(c_{11} cos(\alpha x) + c_{12} sin(\alpha x))$ Applying BC #1: $(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y))(c_{11}) = 0$ Since $(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y)) \neq 0$, hence, $c_{11} = 0$ $\rightarrow u_3 = (c_9 cosh(\alpha y) + c_{10} sinh(\alpha y))(c_{12} sin(\alpha x))$ Applying BC #2: $(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y))(c_{12} sin(\alpha)) = 0$ Since $(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y)) \neq 0$, $c_{12} \neq 0$ *when* $sin(\alpha) = 0$ for $\alpha = n\pi$, where $n = 1,2,3 \ldots$ There are infinite solutions in Case 3: $u_{3,n} = (c_{9,n} cosh(n\pi y) + c_{10,n} sinh(n\pi y))(c_{12,n} sin(n\pi x))$ $u_{3,n} = (A_{3,n} cosh(n\pi y) + B_{3,n} sinh(n\pi y))(sin(n\pi x))$ *where* n *=1,2,3,…*

In summary, the eigenvalue and eigenfunction of the PDE for each case are listed below:

Case	PDE solution	Eigenvalue and eigenfunction of PDE
Case #1:	$u_1 = 0$	No solution

(λ=0)		hence no eigenvalue and no eigenfunction
Case #2: (λ= $-\alpha^2$) $\alpha > 0$	$u_2 = 0$	No solution hence no eigenvalue and no eigenfunction
Case #3: (λ= $+\alpha^2$) $\alpha > 0$	$u_{3,n} = \left(A_{3,n}cosh(n\pi y) + B_{3,n}sinh(n\pi y)\right)(sin(n\pi x))$ *where* $n = 1,2,3,\ldots$	*Eigenvalue,* $\lambda_n = +\alpha_n{}^2 = (n\pi)^2$ *Eigenfunction* $u_{3,n} = \left(A_{3,n}cosh(n\pi y) + B_{3,n}sinh(n\pi y)\right)(sin(n\pi x))$

Step 4: Superposition Principle to find $u_{total}(x,y) = X_1Y_1 + X_2Y_2 + X_3Y_3$

$$u_{total}(x,y) = \underbrace{\sum_{n=1}^{\infty}\left(A_{3,n}cosh(n\pi y) + B_{3,n}sinh(n\pi y)\right)(sin(n\pi x))}_{solution\ from\ Case\ 3}$$

Expanding it, we obtain

$$u_{total}(x,y) = \sum_{n=1}^{\infty}\left(A_{3,n}cosh(n\pi y)(sin(n\pi x))\right) + \sum_{n=1}^{\infty}\left(B_{3,n}sinh(n\pi y)\ (sin(n\pi x))\right)$$

where there are 2 remaining unknowns ($A_{3,n}$ & $B_{3,n}$).

Step 5: Continue to apply the remaining BC & Fourier series expansion.

BC #3: $u(x,\ 0) = 0$ for $0 < x < a$

$$u_{total}(x,0) = \sum_{n=1}^{\infty}\left(A_{3,n}(sin(n\pi x))\right) = 0$$

Recall Half-range Fourier Sine Series Expansion:

$f(x) = \sum_{n=1}^{\infty}(b_n \sin n\omega x)$ *where* $b_n = \frac{2}{L}\int_0^{\tau} f(x)\sin n\omega x\, dx$

We notice $f(x) = 0$; $A_{3,\text{n}} = \frac{2}{L}\int_0^{\tau} f(x)\sin n\omega x\, dx = 0$

$$\rightarrow u_{total}(x,y) = \sum_{n=1}^{\infty}\left(B_{3,n}sinh(n\pi y)\ (sin(n\pi x))\right)$$

BC #4: $u(x,\ 1) = 100$ for $0 < x < 1$

$$u_{total}(x,1) = \sum_{n=1}^{\infty}\left(B_{3,n}sinh(n\pi)\ (sin(n\pi x))\right) = 100$$

Recall Half-range Fourier Sine Series Expansion:

$$f(x) = \sum_{n=1}^{\infty}(b_n \sin n\omega x)$$
$$\text{where } b_n = \frac{2}{L}\int_0^{\tau} f(x) \sin n\omega x \, dx$$

We notice $B_{3,n} sinh(n\pi) = \frac{2}{L}\int_0^{\tau} f(x) \sin n\omega x \; dx$

where angular frequency, $\omega = \pi$, period, $p = \frac{2\pi}{\omega} = 2$, $f(x) = 100$

Finite interval, τ=1, half period, $L = \frac{p}{2} = \frac{2}{2} = 1$

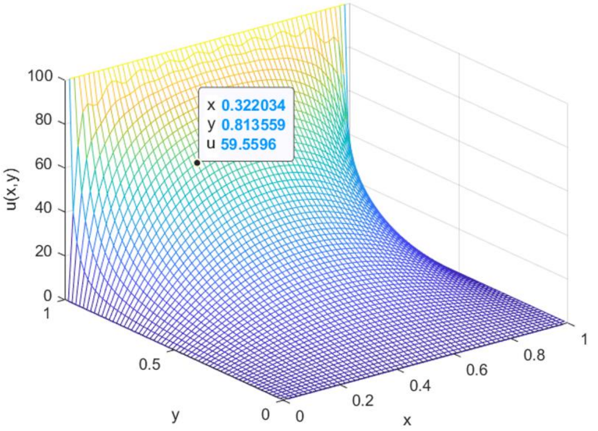
$$B_{3,n} sinh(n\pi) = \frac{2}{1}\int_0^{1} 100 \sin n\omega x \, dx$$
$$B_{3,n} = \frac{200}{sinh(n\pi)}\int_0^{1} \sin n\omega x \, dx = \frac{200}{sinh(n\pi)}\left[\frac{-\cos n\pi - (-1)}{n\pi}\right] = \frac{200}{n\pi sinh(n\pi)}[1 - (-1)^n]$$

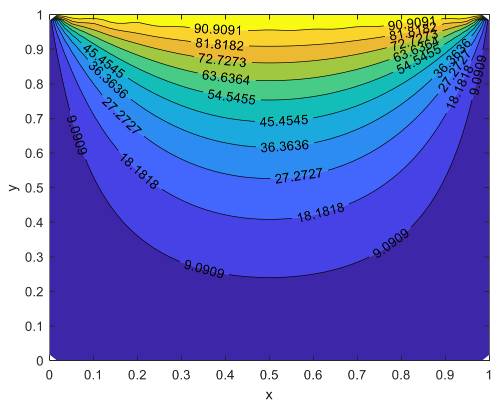
Thus, we have solved all the unknowns and obtain the particular PDE solution:

$$\therefore u_{total}(x, y) = \sum_{n=1}^{\infty}\Big(B_{3,n} sinh(n\pi y)\,(sin(n\pi x))\Big)$$

$$u_{total}(x, y) = \sum_{n=1}^{\infty}\left(\frac{200}{n\pi sinh(n\pi)}[1 - (-1)^n] sinh(n\pi y)\,(sin(n\pi x))\right)$$

We can use the PDE solution to estimate the temperature distribution at any point. Example: The temperature results at 60×60 points of the (x, y) locations have been plotted below:

<table>
<tr><th>3D plot of $u_{total}(x, y)$ wrt x- & y- axis</th><th>2D plot contour plot</th></tr>
<tr><td>


-Due to boundary conditions on 4 sides of the plates, the temperature becomes stable after certain period.
</td><td>


-Top view of the 3D plot with the contour, i.e. line with same magnitude.

Try to find the location of the plate that has temperature around 60 ^oC.
</td></tr>
</table>

For example, $u_{total}(0.322,0.814) \approx 60\ ^oC$. Note that 20 terms are used for plotting the graphs. For higher accuracy, more terms & more grids can be included but computational time will be increased. Try to verify the answer: $u_{total}(0.322,0.814)$ $\approx \sum_{n=1}^{20}\left(\frac{200}{n\pi sinh(n\pi)}[1-(-1)^n]sinh(0.814n\pi)\ (sin(0.322n\pi))\right)$	Hint: Green color

13.3 SOLVING NON-HOMOGENEOUS BOUNDARY CONDITION VIA SUPERPOSITION PRINCIPLE

A Dirichlet problem for a rectangle can be readily solved by separation of variables when homogeneous boundary conditions are specified on two parallel boundaries. However, the method of separation variables is not applicable to a Dirichlet problem when the boundary conditions on all four sides of the rectangle are non-homogeneous. For example,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \qquad 0<x<a, \quad 0<y<b$$

$$u(0,y)=F(y), \qquad u(a,y)=G(y), \qquad 0<y<b$$

$$u(x,0)=f(x), \qquad u(x,b)=g(x), \qquad 0<x<a$$

The general PDE solution remains the same as below:

$$u(x,y)=\underbrace{(c_1+c_2y)(c_3+c_4x)}_{Solution\ of\ Case\ 1}+\underbrace{(c_5cos(\alpha y)+c_6sin(\alpha y))\,(c_7cosh(\alpha x)+c_8sinh(\alpha x))}_{Solution\ of\ Case\ 2}$$

$$+\underbrace{(c_9cosh(\alpha y)+c_{10}sinh(\alpha y))\,(c_{11}cos(\alpha x)+c_{12}sin(\alpha x))}_{Solution\ of\ Case\ 3}$$

To apply the following boundary conditions:

Boundary condition (BC) #1: $u(0,y)=F(y)$, BC #2: $u(a,y)=G(y)$

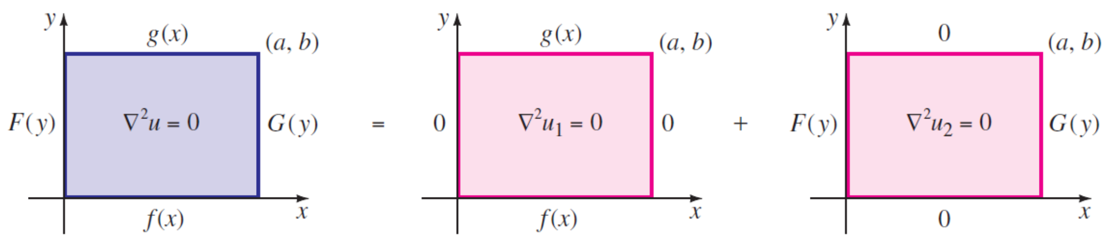
Case	Applying BC #1 & BC #2 or (BC #3 & BC #4 return same result)
Case #1: (λ=0)	$u_1=X_1(x)Y_1(y)$ $=(c_1+c_2y)(c_3+c_4x)$ Applying BC #1, $(c_1+c_2y)(c_3)=F(y)$

	Applying BC #2, $(c_1 + c_2 y)(c_4 a) = G(y)$ Since no unique $c_1 - c_4$ can be obtained via the BC #1-#4, no particular solution can be obtained. This is due to non-homogeneous BC.
Case #2: $(\lambda = -\alpha^2)$ $\alpha > 0$	$u_2 = X_2(x)Y_2(y)$ $= (c_5 cos(\alpha y) + c_6 sin(\alpha y))(c_7 cosh(\alpha x) + c_8 sinh(\alpha x))$ Applying BC #1: $(c_5 cos(\alpha y) + c_6 sin(\alpha y))(c_7) = F(y)$ Applying BC #2: $(c_5 cos(\alpha y) + c_6 sin(\alpha y))(c_7 \cosh(\alpha a) + c_8 \sinh(\alpha a)) = G(y)$ Since no unique $c_5 - c_8$ can be obtained via the BC #1-#4, no particular solution can be obtained. This is due to non-homogeneous BC.
Case #3: $(\lambda = +\alpha^2)$ $\alpha > 0$	$\therefore u_3 = X_3(x)Y_3(y)$ $= (c_9 cosh(\alpha y) + c_{10} sinh(\alpha y))(c_{11} cos(\alpha x) + c_{12} sin(\alpha x))$ Applying BC #1: $(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y))(c_{11}) = F(y)$ Applying BC #2: $(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y))(c_{11} cos(\alpha a) + c_{12} sin(\alpha a)) = G(y)$ Since no unique $c_9 - c_{12}$ can be obtained via the BC #1-#4, no particular solution can be obtained. This is due to non-homogeneous BC.

In the previous examples, homogenous BC can ensure the unique particular solution of a boundary value problem to exist. However, it is difficulty to get the solution directly if non-homogeneous BC is encountered. The PDE problem with non-homogeneous can be solved if it can be separated into sub-problems with homogeneous BC. For example,

Sub-problem #1 with homogeneous BC:	Sub-problem #2 with homogeneous BC:
$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0, \quad 0 < x < a, 0 < y < b$ $u_1(0, y) = 0, u_1(a, y) = 0, \quad 0 < y < b$ $u_1(x, 0) = f(x), u_1(x, b) = g(x), 0 < x < a$	$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0, \quad 0 < x < a, 0 < y < b$ $u_2(0, y) = F(y), u_2(a, y) = G(y), 0 < y < b$ $u_2(x, 0) = 0, u_2(x, b) = 0, \quad 0 < x < a$

As shown in the figure below, PDE due to non-homogeneous PDE can be solved by separating it into two sub-problems, where the solutions of sub-problem #1, $u_1(x, y)$ and sub-problem #2, $u_2(x, y)$ can be added in the superposition manner to obtain the total solution, $u(x, y)$.



Note: ∇^2 is called Laplacian or Laplace operator. For 2D problem, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

In this way, *u* satisfies all boundary conditions in the original problem:

$$u(x,y) = u_1(x,y) + u_2(x,y)$$

For example,

Solution of sub-problem 1:

$$u_1(x,y) = \sum_{n=1}^{\infty}\left\{A_n \cosh\frac{n\pi}{a}y + B_n \sinh\frac{n\pi}{a}y\right\}\sin\frac{n\pi}{a}x$$

where

$$A_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi}{a}x\,dx \quad ; \quad B_n = \frac{1}{\sinh\frac{n\pi}{a}b}\left(\frac{2}{a}\int_0^a g(x)\sin\frac{n\pi}{a}x\,dx - A_n\cosh\frac{n\pi}{a}b\right)$$

Solution of sub-problem 2:

$$u_2(x,y) = \sum_{n=1}^{\infty}\left\{C_n \cosh\frac{n\pi}{b}x + D_n \sinh\frac{n\pi}{b}x\right\}\sin\frac{n\pi}{b}y$$

where

$$C_n = \frac{2}{b}\int_0^a F(y)\sin\frac{n\pi}{b}y\,dy \quad ; \quad D_n = \frac{1}{\sinh\frac{n\pi}{b}a}\left(\frac{2}{b}\int_0^a G(y)\sin\frac{n\pi}{b}y\,dy - C_n\cosh\frac{n\pi}{b}a\right)$$

Total Solution of original problem:

$$u(x,y) = \sum_{n=1}^{\infty}\left\{A_n \cosh\frac{n\pi}{a}y + B_n \sinh\frac{n\pi}{a}y\right\}\sin\frac{n\pi}{a}x + \sum_{n=1}^{\infty}\left\{C_n \cosh\frac{n\pi}{b}x + D_n \sinh\frac{n\pi}{b}x\right\}\sin\frac{n\pi}{b}y$$

where

$$A_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi}{a}x\,dx \quad ; \quad B_n = \frac{1}{\sinh\frac{n\pi}{a}b}\left(\frac{2}{a}\int_0^a g(x)\sin\frac{n\pi}{a}x\,dx - A_n\cosh\frac{n\pi}{a}b\right)$$

$$C_n = \frac{2}{b}\int_0^a F(y)\sin\frac{n\pi}{b}y\,dy \quad ; \quad D_n = \frac{1}{\sinh\frac{n\pi}{b}a}\left(\frac{2}{b}\int_0^a G(y)\sin\frac{n\pi}{b}y\,dy - C_n\cosh\frac{n\pi}{b}a\right)$$