STRATEGIES TO SOLVE FIRST ORDER DIFFERENTIAL EQUATIONS

WEEK 2: STRATEGIES TO SOLVE FIRST ORDER DIFFERENTIAL EQUATIONS

2.1 VERIFICATION OF DIFFERENTIAL EQUATION'S SOLUTION

A solution to an ODE is a function which is differentiable, and which satisfies the given equation. This is true for both explicit and implicit solutions.

Case (1): Explicit solution
An *explicit solution* is any solution that is given in the form of $y = Q(t)$ where y is the dependent variable; $Q(t)$ is the function of independent variable. The explicit solution is valid in the interval $I: \alpha < t < \beta$ if the following conditions are meet: (i) $Q(t), \frac{dQ(t)}{dt}, \dots, \frac{d^{n-1}Q(t)}{dt^{n-1}}$ is differentiable (ii) $Q(t)$ can satisfy the differential equation

Case (2): Implicit solution
An implicit solution is any solution that isn't in explicit form ($y = Q(t)$).

Example 2.1:

Find implicit and explicit solution to the first order differential equation $y\frac{dy}{dt} = t$, $y(2) = -1$.

Solution:

$y\frac{dy}{dt} = t$

>> $\int ydy = \int tdt$

>> $\frac{y^2}{2} = \frac{t^2}{2} + C$ [***Comment***: *This is general implicit solution*]

Applying the initial condition $y(2) = -1$, we get

>> $\frac{(-1)^2}{2} = \frac{(2)^2}{2} + C$

>> $C = -\frac{3}{2}$

$\therefore y^2 = t^2 - 3$ [***Comment***: *This is particular implicit solution*]

Generally, we arrange the implicit solution in the form *G*(*t*,*y*)=0, i.e. $y^2 - t^2 = -3$.

Rearrange the implicit solution and let LHS to be the dependent variable, we get

>> $y = \pm\sqrt{t^2 - 3}$ [***Comment***: *This is particular explicit solution*]

In this case, it is found that there are two explicit solutions:

>> $y_1 = +\sqrt{t^2 - 3}$

>>$y_2 = -\sqrt{t^2 - 3}$

To check which one is the true solution, reapply the initial condition, $y(2) = -1$.

>>> $y_1 = +\sqrt{t^2 - 3} = +\sqrt{2^2 - 3} = 1$ [***Comment:*** *This is not the true explicit solution*]

>>> $y_2 = -\sqrt{t^2 - 3} = -\sqrt{2^2 - 3} = -1$ [***Comment***: *This is the true explicit solution*]

Verification of solution:

(i) To verify the $y^2 = t^2 - 3$ is the ***implicit solution*** for the differential equation $\frac{dy}{dt} = t$. We try to deduce the differential equation from it.

For example:

$y^2 = t^2 - 3$

>> $\frac{d}{dt}[y^2] = \frac{d}{dt}[t^2 - 3]$

>> $2y\frac{dy}{dt} = 2t$

>> $y\frac{dy}{dt} = t$ [**Comment:** Satisfy the original ODE problem]

$\therefore$ Thus, it is proven that $y^2 = t^2 - 3$ is the *implicit solution* for the $y\frac{dy}{dt} = t$.

(ii) To verify the $y_2 = -\sqrt{t^2 - 3}$ is the ***explicit solution*** for the differential equation $\frac{dy}{dt} = t$. We differentiate and substitute it to the equation.

For example:

$y_2 = -\sqrt{t^2-3}$

$>> \frac{d}{dt}[y_2] = \frac{d}{dt}[-\sqrt{t^2-3}]$

$>> \frac{d}{dt}[y_2] = -\frac{1}{2}(t^2-3)^{\frac{1}{2}-1}.2t]$

$>> \frac{d}{dt}[y_2] = -\frac{t}{\sqrt{(t^2-3)}}$

Substitute the derivative to the differential equation $y\frac{dy}{dt} = t$

LHS we get,

$y\frac{dy}{dt} = y_2\left(-\frac{t}{\sqrt{(t^2-3)}}\right)$

Substitute $y_2 = -\sqrt{t^2-3}$ into the equation, we get

$y\frac{dy}{dt} = -\sqrt{t^2-3}\left(-\frac{t}{\sqrt{(t^2-3)}}\right)$

$>> y\frac{dy}{dt} = t$ [**Comment:** Satisfy the original ODE problem]

∴ Since LHS=RHS, thus it is proven that $y_2 = -\sqrt{t^2-3}$ is the *explicit solution* for the differential equation $y\frac{dy}{dt} = t$.

(iii) **Exercise**: Try to prove if $y_1 = +\sqrt{t^2-3}$ is an ***explicit solution*** for the differential equation if the initial condition is given as $y(2) = -1$.

Note: In some cases, we might be able to obtain the *implicit solution* but we might not able to get the explicit solution.

Example 2.2

For differential equation $3y^3e^{3xy} - 1 + (2ye^{3xy} + 3xy^2e^{3xy})\frac{dy}{dx} = 0$, initial condition, $y(0) = 1$. Given that $y^2e^{3xy} - x = 1$ is the implicit solution to the differential equation. Please verify the solution.

Verification of the implicit solution:

$y^2e^{3xy} - x = 1$

$>> \frac{d}{dx}[y^2e^{3xy}] - \frac{d}{dx}[x] = \frac{d}{dx}[1]$

$>> \frac{d}{dx}[y^2].e^{3xy} + y^2.\frac{d}{dx}[e^{3xy}] - 1 = 0$

$>> 2y\frac{dy}{dx}.e^{3xy} + y^2.\frac{d}{dx}[3xy].e^{3xy} - 1 = 0$

$>> (2ye^{3xy})\frac{dy}{dx} + y^2.(\frac{d}{dx}[3x].y + 3x.\frac{d}{dx}[y]).e^{3xy} - 1 = 0$

$>> (2ye^{3xy})\frac{dy}{dx} + y^2.(3y+3x\frac{dy}{dx}).e^{3xy} - 1 = 0$

$>> (2ye^{3xy})\frac{dy}{dx} + (3y^3+3xy^2\frac{dy}{dx}).e^{3xy} - 1 = 0$

$>> 3y^3e^{3xy} - 1 + (2ye^{3xy} + 3xy^2e^{3xy})\frac{dy}{dx} = 0$ [**Comment:** Satisfy the original ODE problem]

∴ The *implicit solution* is proven

Observation: There is no way to rearrange the implicit solution for $y = y(x)$ and get an *explicit solution*. This mostly happen for nonlinear case.

2.2 STRATEGY TO SOLVE 1ST ORDER DIFFERENTIAL EQUATION

There are many strategies that have been developed to solve differential equation. We will start with the most fundamental one. Bear in your mind various strategies can be implemented depends on the types/forms of differential equation.

First of all, we will start with the strategies to solve 1st order ***linear*** differential equation, i.e. $a_1(x)y' + a_0(x)y = g(x)$. These strategies include

(a) ***exact differential equation*** [*Appendix 2.1*]

(b) ***linear differential equation*** [*Section 2.2.1*]

(c) ***separable differential equation*** [*Section 2.2.2*]

Note 1: In this study, we will focus in learning the strategies (b) and (d) only to solve 1st order linear ODE with constant coefficient.

Moreover, you can check the appendix to find more strategies used for solving the 1st order ***nonlinear*** differential equation, i.e. $a_1(x,y)(y')^C + a_0(x,y)(y)^D = g(x,y)$. These strategies include

(d) ***separable differential equation*** [*Section 2.2.2*]

(e) ***Bernoulli's equation*** [*Appendix 2.2*]

(f) ***homogeneous differential equation*** [*Appendix 2.3*]

(g) ***nonhomogeneous differential equation*** [*Appendix 2.4*]

Note 2: In many cases there are at least one of these strategies which can be used to solve the problem. The appendix serves as the extra materials for your extra knowledge.

2.2.1 LINEAR DIFFERENTIAL EQUATION

For most of the 1st order linear differential equation, it can be solved by using the following procedure.

Note 1: Integrating factor is a function that is chosen to allow an "inexact" differential equation to be made into an "exact" differential equation.

Note 2: A simple example is given to illustrate that Exact differential equation can be solved directly.

Exact Differential Equation	Inexact Differential Equation
$e^{y^2}\frac{dx}{dy} + e^{y^2}(2y)x = 0$ [*Exact*] $\frac{d}{dy}\left(x.e^{y^2}\right) = 0$ **Hint:** Product rule $\frac{d}{dy}\left(x.e^{y^2}\right) = e^{y^2}\frac{dx}{dy} + e^{y^2}(2y)x$	$\frac{dx}{dy} + (2y)x = 0$ [*Inexact*] **Hint:** Can't be solved directly, thus we *convert it to exact differential equation by multiplying it with integrating factor*, i.e. e^{y^2}

Procedure to solve 1st linear ODE problem using linear differential equation

Step 1: Arrange the differential equation in the *linear form* of

$$\frac{dy}{dx} + p(x)y = q(x)$$

where $x = independent\ variable$ and $y = dependent\ variable$

Step 2: Create *integrating factor*,

$$IF = e^{\int p(x)dx}$$

Step 3: Multiply 1st ODE eqn by IF

$$IF.\left(\frac{dy}{dx}\right) + IF.[p(x)y] = IF.[q(x)]$$

Step 4: Recognize the LHS is exact solution,

i.e. LHS: $IF.\left(\frac{dy}{dx}\right) + IF.[p(x)y] = \frac{d}{dx}(dependent\ variable\ .\ IF)$

Prove: $\frac{d}{dx}(dependent\ variable\ .\ IF)$

$$= \frac{d}{dx}\left(y.e^{\int p(x)dx}\right)$$

$$= \left(\frac{dy}{dx}\right)e^{\int p(x)dx} + y\left(\frac{d}{dx}e^{\int p(x)dx}\right)$$

$$= \left(\frac{dy}{dx}\right)e^{\int p(x)dx} + y\left[\frac{d}{dx}p(x)dx\right]e^{\int p(x)dx}$$

$$= \left(\frac{dy}{dx}\right)e^{\int p(x)dx} + y[p(x)]e^{\int p(x)dx}$$

$$= IF.\left(\frac{dy}{dx}\right) + IF.[p(x)y]$$ [proven

From steps 3 & 4:

$$IF.\left(\frac{dy}{dx}\right) + IF.[p(x)y] = IF.[q(x)]$$

$$\frac{d}{dx}[y.(IF)] = IF.[q(x)]$$

Step 5:

Integrate both sides, $\int \frac{d}{dx}[y.(IF)]dx = \int IF.[q(x)]\ dx$

and obtain the solution in the explicit form of $y = y(x)$

Example 2.3: Solve the differential equation $xy' - y = 0$ by using Linear Differential Equation.

Solution:

Solve $xy' - y = 0$

where $x = independent\ variable$ and $y = dependent\ variable$

>> $\frac{dy}{dx} - \frac{1}{x}y = 0$ [***Step 1***- *linear form of* $\frac{dy}{dx} + p(x)y = q(x)$]

where

$p(x) = -\frac{1}{x}$

$q(x) = 0$

>> The *integrating factor*, $IF = e^{\int -\frac{1}{x}dx} = e^{-lnx} = x^{-1}$ [***Step 2***- *IF*=$e^{\int p(x)dx}$]

>> $x^{-1}(\frac{dy}{dx}) - x^{-1}(\frac{1}{x}y) = x^{-1}(0)$ [***Step 3***- *Multiply*]

>>$x^{-1}(\frac{dy}{dx}) - (\frac{1}{x^2}y) = 0$

>> $\frac{d}{dx}(y.x^{-1}) = 0$ [***Step 4***- *Exact*]

where LHS=$\frac{d}{dx}(dependent\ variable.IF)$

>>$\int \frac{d}{dx}(x^{-1}y)dx = \int 0dx$ [***Step 5***- *Integrate*]

>> $x^{-1}y = C$

>> $y = Cx$

∴ The solution of the first order linear ODE problem $xy' - y = 0$ is $y = Cx$, where C = arbitrary constant.

Exercise: Verify that the solution is correctly determined.

Example 2.4: Solve the differential equation $\frac{dx}{y} + 2xdy = 0$ by using Linear Differential Equation.

Solution:

$\frac{dx}{y} + 2xdy = 0$

>> $\frac{1}{y} + 2x\frac{dy}{dx} = 0$ [***Comment:*** *This is a nonlinear ODE; Rearrange it to linear ODE*]

[where $x = independent\ variable$ and $y = dependent\ variable$]

Exercise: What is the nonlinear component for the original eqn.?

Rearrange it (Let $x = dependent\ variable$ and $y = independent\ variable$], we get

>> $\frac{dx}{dy} + (2y)x = 0$ [***Step 1****- linear form*]

where

$p(y) = 2y$

$q(y) = 0$

>> The integrating factor, $IF = e^{\int 2ydy} = e^{y^2}$ [***Step 2****- IF*]

>> $e^{y^2}\frac{dx}{dy} + e^{y^2}(2y)x = 0$

[***Step 3***- *Multiply*]

>> $\frac{d}{dy}\left(x.e^{y^2}\right) = 0$

[***Step 4***- *Exact*]

>> $\int \frac{d}{dy}\left(x.e^{y^2}\right)dy = \int 0dy$ [***Step 5***- *Integrate*]

>>$\left(x.e^{y^2}\right) = C$

$\therefore x = Ce^{-y^2}$, where C = arbitrary constant.

Example 2.5 Solve the initial value problem $\frac{dy}{dx} = \frac{y}{2x+3y^2-2}$, $y(1) = 1$

Solution:

$\frac{dy}{dx} = \frac{y}{2x+3y^2-2}$

>> $\frac{dy}{dx} - \frac{y}{2x+3y^2-2} = 0$ [***Comment:*** *This is a nonlinear ODE; Rearrange it to linear ODE*]

Rearrange it (Let $x = dependent\ variable$ and $y = independent\ variable$], we get

>> $\frac{dx}{dy} - \frac{2x+3y^2-2}{y} = 0$

>> $\frac{dx}{dy} - \left(\frac{2}{y}\right)x = \frac{3y^2-2}{y}$ [***Step 1***- *linear form*]

>> The integrating factor, $IF = e^{\int -\left(\frac{2}{y}\right)dy} = e^{-2lny} = y^{-2}$ [***Step 2***- *IF*]

>> $y^{-2}\frac{dx}{dy} - y^{-2}\left(\frac{2}{y}\right)x = y^{-2}\left(\frac{3y^2-2}{y}\right) = \left(\frac{3y^2-2}{y^3}\right)$ [***Step 3***- *Multiply*]

>> $\frac{d}{dy}(x.y^{-2}) = \left(\frac{3y^2-2}{y^3}\right)$ [*Step 4- Exact*]

>> $\int \frac{d}{dy}(x.y^{-2})dy = \int \left(\frac{3y^2-2}{y^3}\right) dy$ [*Step 5- Integrate*]

>> $x.y^{-2} = \int \left(\frac{3}{y}\right) dy + \int \left(\frac{-2}{y^3}\right) dy$

>> $x.y^{-2} = 3lny + y^{-2} + C$,where C = arbitrary constant.

Using initial condition, $y(1) = 1$ (i.e. when $x = 1, y = 1$)

>> $(1).(1)^{-2} = 3\ln(1) + (1)^{-2} + C$

>> $C = 0$

$\therefore x = 3y^2 lny + 1$

In this study, it shows that the *integrating factor,* $IF = e^{\int p(x)dx}$ is very useful in solving 1st order linear ODE. For your additional knowledge, there are other types of integrating factors to solve various ODE problems, as shown in the *Appendix 2.5.*

2.2.2 SEPARABLE DIFFERENTIAL EQUATION

A *separable differential equation* can be written in the form of

$$g(y)dy = f(x)dx$$

where the variables are separable

(**LHS** = function in y variable; **RHS** = function in x variable).

Then, we can solve the problem by integrating both sides. This is particularly useful in solving certain linear & nonlinear differential equation which is separable.

Example 2.6: Solve the equation $\frac{dy}{dx} = \frac{y+1}{x-4}, y(6) = 0$

Solution:

$\frac{dy}{dx} = \frac{y+1}{x-4}$ [***Comment***- *Linear form*]

>> $\int \frac{1}{y+1} dy = \int \frac{1}{x-4} dx$ [***Step 1***- *Separable form & Integrate both sides*]

>> $ln|y+1| = ln|x-4| + C$

Apply initial condition, $y(6) = 0$ *to solve the unknown* C

>> $ln|0+1| = ln|6-4| + C$

>> $C = -ln|2|$

>> $ln|y+1| = ln|x-4| - ln|2|$

>> $ln|y+1| = ln\frac{|x-4|}{|2|}$

>> $y+1 = \frac{x-4}{2}$

$\therefore y = \frac{x}{2} - 3$

Example 2.7: Solve the equation $9y\frac{dy}{dx} - 4x = 0$.

Solution:

In fact, this is a nonlinear equation that is difficult to be solved by using Linear Differential Equations. However, it can be solved easily by using the Separable Differential Equation.

$9y\frac{dy}{dx} - 4x = 0$ [***Comment***- *Nonlinear form*]

Think: What is the nonlinear component?

>> $9ydy = 4xdx$ [***Step 1***- *Separable form*]

>> $\int 9ydy = \int 4xdx$ [***Step 2***- *Integrate both sides*]

$\therefore \frac{9}{2}y^2 = 2x^2 + C$

Example 2.8: Solve the equation $y' = 1 + y^2$.

Solution:

$\frac{dy}{dx} - y^2 = 1$ [***Comment***- *Nonlinear form*]

>> $\frac{dy}{dx} = 1 + y^2$

>> $\frac{dy}{1+y^2} = dx$ [***Step 1***- *Separable form*]

>> $\int \frac{dy}{1+y^2} = \int dx$ [***Step 2***- *Integrate both sides*]

>> $tan^{-1}y = x + C$

$\therefore y = tan(x + C)$