SOLUTIONS TO HOMOGENEOUS LINEAR 2ND ORDER ODE

WEEK 3: SOLUTIONS TO HOMOGENEOUS LINEAR 2ND ORDER ODE

3.1 INTRODUCTION TO 2ND ORDER ODE

We started the discussion of ordinary differential equations (ODE) with the 1st order ODE. The order of a differential equation is the degree of the highest derivative that occurs in the equation. Based on this definition, a 2nd order ODE has the second order derivative in the differential equation. To give a quick example, $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = 0$ is a 2nd order ODE.

As discussed before, the differential equation is important, because many engineering or physical scenarios have behaviors or responses that are represented (a.k.a. modeled) as ordinary differential equations (ODE). However, not every scenario can be modeled by the 1st order ODE. We use a basic circuit scenario to illustrate this point.

Example 3.1:

<table>
<tr><th>1st order ODE</th><th>2nd order ODE</th></tr>
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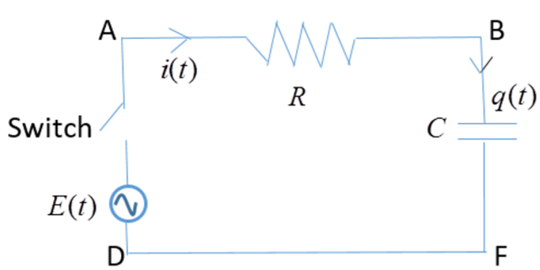
1st order differential equation is sufficient to model an simple electrical circuit with just a resistor and a capacitor in series (known as the RC circuit):



The potential differences are:

Resistor: $\Delta V(t) = (V_B - V_A) = Ri(t) = R\frac{dq(t)}{dt}$

Capacitor: $\Delta V(t) = (V_F - V_B) = \frac{1}{C}q(t)$

where *q(t)* is the charge and *i(t)* is the rate of

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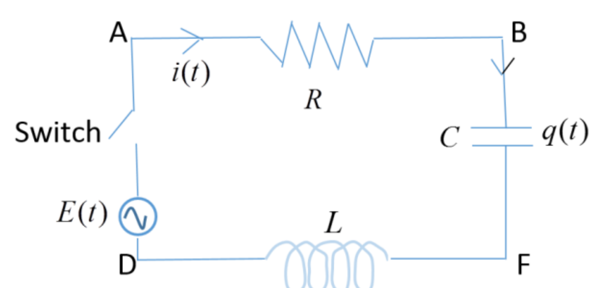
However, if the electrical circuit also contains an inductor in series (known as the RLC circuit), then:



The potential differences are:

Resistor: $\Delta V(t) = (V_B - V_A) = Ri(t) = R\frac{dq(t)}{dt}$

Capacitor: $\Delta V(t) = (V_F - V_B) = \frac{1}{C}q(t)$

Inductor: $\Delta V(t) = (V_D - V_F) = L\frac{di(t)}{dt} = L\frac{d^2q(t)}{dt^2}$

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change of charge which is the current. By Kirchoff's voltage law, considering the single voltage loop, an equation that relates all voltages can be formed: $R\frac{dq(t)}{dt}+\frac{1}{C}q(t)=E(t)$	Again, by Kirchoff's voltage law, the equation that describes this single voltage loop becomes: $L\frac{d^2q(t)}{dt^2}+R\frac{dq(t)}{dt}+\frac{1}{C}q(t)=E(t)$

Extra Info: *According to Kirchhoff's voltage law, the sum of all the voltages around any closed network / loop is equal to zero. In other words, all the voltage drops equal to the input or supplied voltage(s).*

Just as the 1st order ODE, the concepts of homogeneity and linearity of a differential equation apply to 2nd order ODE as well. As a recap, for an nth order ODE:

$$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+\cdots+a_2(x)y''+a_1(x)y'+a_0(x)y=g(x)$$

the ODE is linear if all the coefficients a_n do not contain product with the dependent variable or its derivatives, and the derivatives themselves are of the first power only. Meanwhile, within the context of linear ODE, an ODE is homogeneous if the RHS, $g(x)=0$, and non-homogeneous if $g(x)\neq 0$. The following are some examples of 2nd order ODE that show linearity and homogeneity:

(1) Linear ODE example: $\frac{d^2y}{dx^2}+5x\frac{dy}{dx}+4y=x^2$	**(2) Non-linear ODE example:** $\boldsymbol{y}\frac{d^2y}{dx^2}+5xy\left(\frac{dy}{dx}\right)^{\mathbf{2}}+4y=x^2\,\mathbf{cos}\,\boldsymbol{y}$
(3) Homogeneous linear ODE example: $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+4y=0$	**(4) Non-homogeneous linear ODE example:** $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+4y=\mathbf{cos}\,\boldsymbol{x}$

In this topic, we focus on solving the homogeneous linear 2nd order ODE, especially the type displayed by example (3) above, in which the coefficients of all terms are constants.

3.2 LINEARITY PRINCIPLE AND LINEAR DEPENDENCY OF SOLUTIONS

For a 2nd order ODE, it is usually expected that the complete solution consists of more than one component, and the complete solution is obtained as a linear combination of all the component solutions that satisfy the ODE. Therefore, before we proceed to the strategy of solving 2nd order ODEs, we should get ourselves familiar with the linearity principle and linear dependency of solutions.

The importance of the theory is illustrated in the following example:

It is given that the ODE $\frac{d^2y(x)}{dx^2} - 4\frac{dy(x)}{dx} + 3y(x) = 0$ has two possible solutions, namely $y_1 = e^{3x}$ and $y_2 = e^x$. Since a solution must satisfy the equation, let us verify that these two are indeed possible solutions:

Case (1): Assume the solution is $y_1 = e^{3x}$
Verification: $y_1 = e^{3x}$ >> $\frac{dy_1}{dx} = 3e^{3x}$ >> $\frac{d^2y_1}{dx^2} = 9e^{3x}$ Substitute to LHS $\frac{d^2y(x)}{dx^2} - 4\frac{dy(x)}{dx} + 3y(x) = 9e^{3x} - 4(3e^{3x}) + 3e^{3x} = 0$ >> LHS = RHS = 0 $\therefore y_1 = e^{3x}$ *is proven to be the solution of* $\frac{d^2y(x)}{dx^2} - 4\frac{dy(x)}{dx} + 3y(x) = 0$

Case (2): Assume the solution is $y_2 = e^x$
Verification: $y_2 = e^x$ >> $\frac{dy_2}{dx} = e^x$ >> $\frac{d^2y_2}{dx^2} = e^x$ Substitute to LHS

$\frac{d^2y(x)}{dx^2} - 4\frac{dy(x)}{dx} + 3y(x) = e^x - 4(e^x) + 3e^x = 0$

>> LHS = RHS = 0

$\therefore y_2 = e^x$ *is proven to be the solution of* $\frac{d^2y(x)}{dx^2} - 4\frac{dy(x)}{dx} + 3y(x) = 0$

If we linearly combine $y_1 = e^{3x}$ and $y_2 = e^x$, we obtain $y_c = c_1e^{3x} + c_2e^x$ which interestingly also satisfies the ODE (see Case(3)). Since satisfying the equation is the criterion of being a solution, then $y_c = c_1e^{3x} + c_2e^x$ is a solution to the ODE as well. In fact, *all the three solutions satisfy the ODE and thus they are true solutions.*

Case (3): Assume the solution is $y_c = c_1e^{3x} + c_2e^x$

Verification:

$y_c = c_1e^{3x} + c_2e^x$

>> $\frac{dy_c}{dx} = 3c_1e^{3x} + c_2e^x$

>> $\frac{d^2y_c}{dx^2} = 9c_1e^{3x} + c_2e^x$

Substitute to LHS

$\frac{d^2y(x)}{dx^2} - 4\frac{dy(x)}{dx} + 3y(x) = (9c_1e^{3x} + c_2e^x) - 4(3c_1e^{3x} + c_2e^x) + 3(c_1e^{3x} + c_2e^x) = 0$

>> LHS = RHS=0

$\therefore y_c = c_1e^{3x} + c_2e^x$ *is proven to be the solution of* $\frac{d^2y(x)}{dx^2} - 4\frac{dy(x)}{dx} + 3y(x) = 0$

The complete solution is formed according to Linearity Principle / Principle of Superposition as follows:

Linearity Principle / Principle of Superposition:

If y_1 & y_2 are both solutions

of the homogeneous linear differential equation.

Then so is the solution $y_c = c_1y_1 + c_2y_2$

where c_1 & c_2 are arbitrary constants.

Note 1: So, the solution of the 2nd order ODE $\frac{d^2y(x)}{dx^2} - 4\frac{dy(x)}{dx} + 3y(x) = 0$ is not purely $y_1 = e^{3x}$ or $y_2 = e^x$.

Note 2: If a 2nd order linear ODE is encountered, the complete solution will be equal to $y = c_1y_1 + c_2y_2 = c_1e^{3x} + c_2e^x$.

In conclusion, the **general complementary solution of a homogeneous linear ODE** is equal to

$$y = c_1y_1(x) + c_2y_2(x)$$

where $y_1(x)$ & $y_2(x)$ are known as the linearly independent solutions to the equation.

Linearly dependent vs linearly independent solutions: According to the linear dependency theorem, solutions are independent to one another if their linear combination is zero only when all the constants of proportionality are zero. Otherwise, if their linear combination can be zero with one or more constants of proportionality being non-zero, then these solutions are considered dependent to one another. For example, $y_1 = e^x$ and $y_2 = 2e^x$ are known to be linearly dependent on each other (see table below).

A complete solution should have linearly independent component solutions. If two solutions are linearly dependent, it means these two solutions are redundant and hence they do not represent two independent solutions instead of one. This can be illustrated by the same example: $y = e^x + 2e^x = (1+2)e^x$ (only represent one solution).

We can use the following two methods to *check whether two solutions are linearly independent to each other or not*: (a) the Linear Dependency Theorem itself, and (b) Wronskian Method.

Let x = independent variable; y = dependent variable

	Method 1 **(Linear Dependency Theorem)**
(i) Linearly dependent [*Undesired solutions for ODE*]	$y_1(x)$ & $y_2(x)$ are linearly dependent if $c_1y_1(x) + c_2y_2(x) = 0$ where c_1 & $c_2 \neq 0$ Or in other words, $y_1(x)$ & $y_2(x)$ are proportional to each other
For example: Check if $y_1 = e^x$ & $y_2 = 2e^x$ are linearly dependent or not.	***Solution:*** If $y_1 = e^x$ & $y_2 = 2e^x$ are linearly dependent, $c_1e^x + c_2(2e^x) = 0$, where c_1 & $c_2 \neq 0$.

	It was found that when $c_1 = -2$ & $c_2 = 1$, $c_1 e^x + c_2(2e^x) = -2e^x + 2e^x = 0$ ∴ Thus, $y_1 = e^x$ & $y_2 = 2e^x$ are linearly dependent.
(ii) Linearly independent [*Desired solutions for ODE*]	$y_1(x)$ & $y_2(x)$ are linearly independent if $c_1 y_1(x) + c_2 y_2(x) = 0$ only when c_1 & $c_2 = 0$ Or in other words, $y_1(x)$ & $y_2(x)$ are not proportional to each other
For example: Check if $y_1 = e^x$ & $y_2 = e^{3x}$ are linearly dependent or not.	***Solution:*** If $y_1 = e^x$ & $y_2 = e^{3x}$ are linearly independent, $c_1 e^x + c_2(2e^x) = 0$, only when c_1 & $c_2 = 0$. It was found that only when $c_1 = 0$ & $c_2 = 0$, $c_1 e^x + c_2(e^{3x}) = 0e^x + 0e^{3x} = 0$ ∴ Thus, $y_1 = e^x$ & $y_2 = e^{3x}$ are linearly independent.

Note:

If linearly dependent solutions are obtained,

i.e. y_1 & y_2 are linearly dependent,

we do not obtained a complete solution of

$$y = c_1 y_1(x) + c_2 y_2(x).$$

Thus, extra effort / treatment should be continued

to obtain another solution which is linearly independent,

i.e. y_1 & y_2 are linearly independent.

	Method 2 **(Wronskian, $W(y_1, y_2)$)**
(i) Linearly dependent [*Undesired solutions for ODE*]	$y_1(x)$ & $y_2(x)$ are linearly dependent if $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{vmatrix} = 0$
For example: Check if $y_1 = e^x$ & $y_2 = 2e^x$ are linearly dependent or not.	***Solution:*** $y_1 = e^x$ >> $\frac{dy_1}{dx} = e^x$ $y_2 = 2e^x$ >> $\frac{dy_2}{dx} = 2e^x$ $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{vmatrix} = \begin{vmatrix} e^x & 2e^x \\ e^x & 2e^x \end{vmatrix} = e^x(2e^x) - e^x(2e^x) = 0$ ∴ Thus, $y_1 = e^x$ & $y_2 = 2e^x$ are linearly dependent.
(ii) Linearly independent [*Desired solutions for ODE*]	$y_1(x)$ & $y_2(x)$ are linearly independent if $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{vmatrix} \neq 0$
For example: Check if $y_1 = e^x$ & $y_2 = e^{3x}$ are linearly dependent or not.	$y_1 = e^x$ >> $\frac{dy_1}{dx} = e^x$ $y_2 = e^{3x}$ >> $\frac{dy_2}{dx} = 3e^{3x}$ $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{vmatrix} = \begin{vmatrix} e^x & e^{3x} \\ e^x & 3e^{3x} \end{vmatrix} = e^x(3e^{3x}) - e^x(e^{3x}) = 2e^{4x}$ Since $W(y_1, y_2) \neq 0$ ∴ Thus, $y_1 = e^x$ & $y_2 = e^{3x}$ are linearly independent.

3.3 SOLUTIONS TO HOMOGENEOUS LINEAR ODE WITH CONSTANT COEFFICIENTS

Over the years, scientist and engineer have found that the non-zero solution to 2nd order homogeneous linear ODE with constant coefficients to be an exponential function:

$$\text{Solution of } a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$\text{to be } y(x) = e^{mx}$$

Technically, the zero solution $y(x) = 0$ also satisfies the ODE and is a solution. This is called trivial solution. Generally, when solving an ODE, our intention is only to look for the non-trivial solution(s). Meanwhile, the intuition of $y(x) = e^{mx}$ being a solution to the aforementioned ODE is not hard to see: Since the summation of the derivative terms (LHS) equals to 0 (RHS), the derivatives of $y(x)$ likely have the same function form as $y(x)$. So, $y(x)$ being an exponential function is a logical possibility.

However, this is not a complete solution because *2nd order ODE problem should have 2 linearly independent solutions*. To find the complete solution, we therefore follow the strategy utilizing the characteristic / auxiliary equation.

To solve $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$,

(i) First, form the **characteristic equation**, $am^2 + bm + c = 0$

(ii) Solve the characteristic eqn. and find its roots, m_1 & m_2

-This can be easily obtained by quadratic formula, m_1 & $m_2 = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

(iii) Check if these two roots are:

(a) **Real & distinct root**, $m_1 \neq m_2$

(b) **A pair of complex conjugates roots**, $m_1 = m + i\beta$ & $m_2 = m - i\beta$

(c) **Repeated real root**, $m = m_1 = m_2$

(iv) Check the table below for the complete solution.

Recall: Complex conjugate has same magnitude but opposite sign for the imaginary part. For example, the complex conjugate for a complex number $m_1 = (5 + 6i)$ is $m_2 = (5 - 6i)$.

Prove: To obtain the characteristic equation: $am^2 + bm + c = 0$

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

Assume solution: $y(x) = e^{mx}$ to be solution of the 2nd order ODE

>> Its derivative: $y' = me^{mx}$; $y'' = m^2e^{mx}$

Substitute it into equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, we obtain

>> $a(m^2e^{mx}) + b(me^{mx}) + c(e^{mx}) = 0$

>> $e^{mx}(am^2 + bm + c) = 0$

>> Since $e^{mx} \neq 0$, we obtain $am^2 + bm + c = 0$, i.e. the **characteristic equation.**

The summary of the complete solution are listed below. The detail description will be provided next.

Type of Roots	**(a) Real and distinct roots** m_1 & m_2	**(b) A pair of complex conjugates roots** $m_1 = m + i\beta$ & $m_2 = m - i\beta$	**(c) Repeated real root** $m = m_1 = m_2$
Indicator	$b^2 - 4ac > 0$	$b^2 - 4ac < 0$	$b^2 - 4ac = 0$
Complete solution	$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$	$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ Or $y(x) = e^{mx}(A cos\beta x + B sin\beta x)$ where A & B are arbitrary constants **Note:** Both representations are acceptable, the conversion can be found in Appendix 3.1.	$y(x) = c_1 e^{m_1 x} + c_2 x e^{m_2 x}$
Comment	-Complementary solution in e^{mx} form -No treatment is needed		-Complementary solution in e^{mx} & xe^{mx} form -Treatment is needed to avoid linearly dependent solution.

Hint: Euler formula: $e^{\pm ix} = cosx \pm i(sinx); i = \sqrt{-1} = imaginary$.

Case (a): **Real and distinct roots** $m_1 \neq m_2$
Characteristic equation: $am^2 + bm + c = 0$ **Indicator**: >> $b^2 - 4ac > 0$ *Comment: If $b^2 - 4ac$ is greater than 0, it indicates that the roots, m_1 & m_2 are real and distinct.*

Complete solution: >> $y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ where $m_1 \neq m_2$

Example 3.2: The case for real and distinct roots

$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0$

Let $y(x) = e^{mx}$, we obtain **Characteristic equation**: $m^2 + 2m - 3 = 0$

Indicator: $b^2 - 4ac = 2^2 - 4(1)(-3) = 16$

Since $b^2 - 4ac > 0$, it is the case of real and distinct roots.

Solution of Characteristic equation: $(m - 1)(m + 3) = 0$

>> $m_1 = 1, m_3 = -3$

Complete solution:

$\therefore y(x) = c_1 e^{x} + c_2 e^{-3x}$

Case (b): **A pair of complex conjugates roots** $m_1 = m + i\beta \ \& m_2 = m - i\beta$
Characteristic equation: $am^2 + bm + c = 0$ **Indicator**: >> $b^2 - 4ac < 0$ *Comment: If* $b^2 - 4ac$ *is less than* 0*, it indicates that the roots,* m_1 & m_2 *are a pair of complex conjugates.* **Complete solution**: >> $y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ where $m_1 \neq m_2$;

$m_1 = m + i\beta \ \& m_2 = m - i\beta;$

$i = \sqrt{-1} = imaginary$

or

>> $y(x) = e^{mx}(Acos\beta x + Bsin\beta x)$

where c_1, c_2, A & B are arbitrary constants

Note: In this case, the complete solution can be either $y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ or $y(x) = e^{mx}(Acos\beta x + Bsin\beta x)$. Both answers are acceptable. Check Appendix 3.1 for the conversion.

Example 3.3: The case of a pair of complex conjugates roots:

$4\frac{d^2y}{dx^2} + 16\frac{dy}{dx} + 17y = 0$

Let $y(x) = e^{mx}$, we obtain **Characteristic equation**: $4m^2 + 16m + 17 = 0$

Indicator: $b^2 - 4ac = 16^2 - 4(4)(17) = -16$

Since $b^2 - 4ac < 0$, it is the case of pair of complex conjugates roots

Solution of Characteristic equation:

Completing the square we get

$m^2 + 4m + 17/4 = 0$

>> $(m+2)^2 - 4 + 17/4 = 0$

>> $(m+2)^2 = -1/4$

>> $m_1 = -2 + \frac{1}{2}i, \ m_2 = -2 - \frac{1}{2}i$

Complete solution:

$\therefore y(x) = c_1 e^{(-2+\frac{1}{2}i)x} + c_2 e^{(-2-\frac{1}{2}i)x}$, or alternatively: $y(x) = e^{-2x}\left(Acos\frac{1}{2}x + Bsin\frac{1}{2}x\right)$

Case (c): **Repeated real root** $m = m_1 = m_2$
Characteristic equation: $am^2 + bm + c = 0$ **Indicator**: >> $b^2 - 4ac = 0$ ***Comment:*** *If $b^2 - 4ac$ is equal to 0, it indicates that the roots, m_1 & m_2 are repeated real root.* **Complete solution**: *Previous solution is not valid here* >> $y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ where $m_1 = m_2$ and thus $c_1 e^{m_1 x}$& $c_2 e^{m_2 x}$ are linearly dependent solution (undesired situation) in this case. Treatment is needed as follows: >> $y(x) = c_1 e^{m_1 x} + c_2 x e^{m_2 x}$ where x is multiplied to one component solution so that $c_1 e^{m_1 x}$ & $c_2 x e^{m_2 x}$ are linearly independent solution (desired situation).

Example 3.4: The case of a pair repeated roots

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

Let $y(x) = e^{mx}$, we obtain **Characteristic equation**: $m^2 - 4m + 4 = 0$

Indicator: $b^2 - 4ac = (-4)^2 - 4(1)(4) = 0$

Since $b^2 - 4ac < 0$, it is the case of repeated real root.

Solution of Characteristic equation:

$m^2 - 4m + 4 = 0$

>> $(m - 2)(m - 2) = 0$

>> $m_1 = 2,\ m_2 = 2$

Complete solution:

$\therefore y(x) = c_1 e^{2x} + c_2 x e^{2x}$

Overall comment:

1. Suppose that the roots of the characteristic equation are m_1 & m_2, then $e^{m_1 x}$ & $e^{m_2 x}$ are the solution of the differential equation.
2. Since $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ is a linear homogeneous equation, by using the linear independency and linearity principle, the general solution must be
   - (i) $y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ for ***real and distinct root***, $m_1 \neq m_2$.
   - (ii) $y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ or $y(x) = e^{mx}(A\cos\beta x + B\sin\beta x)$ for ***a pair of complex conjugate roots***, $m_1 = m + i\beta$ & $m_2 = m - i\beta$.
   - (iii) However, if there is ***repeated real roots***, $m = m_1 = m_2$, we get linearly dependent solution $y(x) = c_1 e^{mx} + c_2 e^{mx}$ as proven earlier. In this case, $y(x) = xe^{mx}$ is proven as one of the solution of 2nd order ODE and it is linearly independent with e^{mx}. Thus a complete solution with treatment$y(x) = c_1 e^{m_1 x} + c_2 x e^{m_2 x}$ is obtained which satisfy the linearly independency property.

Extra Info: *Check Appendix 3.2 to find the strategy to solve the Homogeneous linear differential equation with non-constant coefficients x^2, ax (Known as Euler-Cauchy Differential Equation).*

$x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0$ [***Strategy:*** *Let solution to be $x = e^t$ & convert to (i)*]