SOLUTIONS TO NON-HOMOGENEOUS LINEAR 2ND ORDER ODE

WEEK 4: SOLUTIONS TO NON-HOMOGENEOUS LINEAR 2ND ORDER ODE

4.1 SOLUTIONS TO NON-HOMOGENEOUS LINEAR ODE WITH CONSTANT COEFFICIENTS

So far, we have discussed the strategy to solve homogeneous problem, now we will continue with the non-homogeneous problem, i.e. $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=r(x)$, where $r(x)\neq 0$. As before, this refers to 2nd order ODE with constant coefficients.

When we solve a homogeneous linear 2nd order ODE, $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$, the solution is named as the **complementary solution**, $y=y_c$. Naturally, we might think that solving the non-homogeneous $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=r(x)$ will give just a different solution $y=y_p$ (which is known as the **particular solution**). However, actual responses from systems modeled as non-homogeneous ODE often clearly display a combination of two parts: a transient part and a steady-state part. It turns out that the solution to $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=r(x)$ is made up of the complementary solution and the particular solution, which means $y=y_c+y_p$.

This can be understood clearer by seeing the equation as:

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0+r(x)$$

so the component y_c, when substituted back the equation, satisfies the RHS of 0, while the component y_p, when substituted back to the equation, satisfies the RHS of $r(x)$:

Using the notation $y''=\frac{d^2y}{dx^2}$ $\quad y'=\frac{dy}{dx}$ $\quad y=y_c+y_p$

$LHS=ay''+by'+cy$

$=a(y_c''+y_p'')+b(y_c'+y_p')+c(y_c+y_p)$

$=(ay_c''+by_c'+cy_c)+(ay_p''+by_p'+cy_p)$

$=(\quad 0 \quad)+(\quad r(x) \quad)=r(x)=RHS$

So $y=y_c+y_p$ is indeed the solution for $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=r(x)$.

4.2 METHOD OF UNDETERMINED COEFFICIENTS

As an overview, solving $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = r(x)$ involves (1) finding y_c and (2) finding y_p:

(1) Solve $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ just as in homogeneous ODE topic → obtain y_c
(2) y_p ← Obtained by the **method of undetermined coefficients**

A general way of understanding the **method of undetermined coefficients** is that the particular solution y_p follows the same form as $r(x)$. In scenarios represented by ODE, $r(x)$ is the input to the system, while the solution y is the output or response. So, in general, the output follows the same form as the input (e.g. a sinusoidal force acting on a spring-loaded mass makes the mass to oscillate sinusoidally).

If the RHS components, $r(x)$ are in the simple form of exponential, polynomial, sine and cosine functions, we can implement the **method of undetermined coefficient** by letting the RHS components to be equal to $e^{\alpha x}P_n(x)$ as following:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = e^{\alpha x}P_n(x)$$

where $P_n(x)$ is a polynomial function of degree n

Hence, we can propose the possible particular solution of $y_p = e^{\alpha x}Q_n(x)$. With this proposed y_p, the remaining task is to substitute y_p and its derivatives back to the ODE and compare between LHS and RHS to determine all the unknown coefficient values. Finally, $y = y_c + y_p$.

The general procedure to solve the 2nd order nonhomogeneous linear ODE using the method of undetermined coefficients is summarized:

General procedure for the method of undetermined coefficient:

2^{nd} order non-homogeneous linear ODE:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = r(x)$$

Step 1: Solve the homogenous part first

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

Depending on the characteristic roots, complementary solution:

$y_c(x) = c_1e^{m_1x} + c_2e^{m_2x}$ or $y_c(x) = c_1e^{m_1x} + c_2xe^{m_2x}$

Step 2: Solve the non-homogeneous part next

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = e^{\alpha x}P_n(x)$$

Possible particular solution:

$$y_p = e^{\alpha x}Q_n(x)$$

where $Q_n(x) = general\ polynomial\ with\ same\ degree\ of\ n\ with\ P_n(x)$,

e.g. $P_2(x) = 5x^2\ where\ n = 2, then\ Q_2 = Ax^2 + Bx + C$

Step 3: If y_p & y_c are linearly dependent,

give treatment / cure to y_p to obtain linearly independent solution.

Proposed particular solution after cure:

$y_p = xe^{\alpha x}Q_n(x)$ or $y_p = x^2e^{\alpha x}Q_n(x)$

Step 4: Solve the undetermined coefficient of the particular solution

by comparing the coefficient on both sides of the equation.

Step 5: The total solution for the *2^{nd} order non-homogeneous linear ODE:*

$$y_{total} = y_c + y_p$$

Note 1: Always solve the complementary solution first before proposing the particular solution.

Note 2: The detail description for the '***Step 1:*** *Solve the homogenous part first*' can be found in the previous section. Now, we will discuss on the '***Step 2:*** *Solve the non-homogeneous part next*'.

The method of undetermined coefficient is only applicable to 2nd order non-homogeneous linear ODE, where the RHS component, $r(x)$ is restricted for exponential, polynomial, sine and cosine functions, i.e. $e^{\alpha x} P_n(x)$.

The exponential function is related directly to the $e^{\alpha x}$ and polynomial function is related directly to the $P_n(x)$. Moreover, the exponential function, $e^{\alpha x}$ is related indirectly to sine & cosine functions through Euler's Formula: $e^{\pm ix} = cosx \pm i(sinx)$.

Example 4.1:

$e^{-i(10x)} = \cos(10x) - isin(10x)$.

Thus, imaginary part of $e^{-i(10x)}$, i.e. $Im\left[e^{-i(10x)}\right] = -\sin(10x)$

Real part of $e^{-i(10x)}$, i.e. $Re(e^{-i(10x)}) = \cos(10x)$

Exercise: What is the imaginary part and real part of $e^{i(5x)}$?

Depends on the RHS function, the possible particular solution is proposed for the 2nd order non-homogeneous linear ODE as shown in table below.

RHS function	The form of $r(x) = e^{\alpha x}P_n(x)$	Possible Particular Solution $y_p = e^{\alpha x}Q_n(x)$	Comment
(i) Pure Exponential Function, e.g. $r(x) = e^{-3x}$	$r(x) = e^{-3x}P_0(x)$ where $\alpha = -3$ & $P_n(x) = 1$ with degree $n = 0$	$y_p = e^{-3x}Q_O$ $= Ae^{-3x}$ where $Q_O =$ general polynomial with degree $n = 0$	Sine, Cosine, Exponential functions are related to each other in Euler formula: $e^{\pm ix} = cosx \pm i(sinx)$, thus they can be represented by the *exponential function or trigonometric function*. The possible particular solution for the RHS sine and cosine functions by let $y_p = Ccosx + Dsinx$. *Both option 1 & 2 are acceptable in this study.*
(ii) Pure Sine Function, e.g. $r(x) = sin5x$ Given $e^{i(5x)}$ $= cos5x + i(sin5x)$ $Im(e^{i(5x)}) = sin5x$	$r(x) = Im(e^{(5i)x})P_0(x)$ where $\alpha = 5i$ & $P_n(x) = 1$ with degree $n = 0$	**Option 1:** $y_p = e^{(5i)x}Q_O$ $= Ae^{5ix}$ $y_{p,actual} = Im(y_p)$ **Option 2:** $y_p = Ccos5x + Dsin5x$	
(iii) Pure Cosine Function, e.g. $r(x) = cos6x$ Given $e^{i(6x)}$ $= cos6x + i(sin6x)$ $Re(e^{i(6x)}) = cos6x$	$r(x) = Re(e^{(6i)x})P_0(x)$ where $\alpha = 6i$ & $P_n(x) = 1$ with degree $n = 0$	**Option 1:** $y_p = e^{(6i)x}Q_O$ $= Ae^{6ix}$ $y_{p,actual} = Re(y_p)$ **Option 2:** $y_p = Ccos6x + Dsin6x$	
(iv) Mixture of Exponential & Cosine Function, e.g. $r(x) = e^{-3x}cos6x$ Given $e^{i(6x)}$ $= cos6x + i(sin6x)$ $Re(e^{i(6x)}) = cos6x$	$r(x)$ $= Re(e^{(6i)x})(e^{(-3x)})P_0(x)$ $= Re(e^{(6i-3)x})P_0(x)$ where $\alpha = 6i - 3$ & $P_n(x) = 1$ with degree $n = 0$	**Option 1:** $y_p = e^{(6i-3)x}Q_O$ $= Ae^{(6i-3)x}$ $y_{p,actual} = Re(y_p)$ **Option 2:** y_p $= e^{-3x}(Ccos6x$ $+ Dsin6x)$	

RHS function	The form of $r(x) = e^{\alpha x}P_n(x)$	Possible Particular Solution $y_p = e^{\alpha x}Q_n(x)$	Comment
(i) Pure Polynomial Function, e.g. $r(x) = 6x^3 + 4x^2 + 5$	$r(x) = e^{(0x)}P_3(x)$ Where $\alpha = 0$ & $P_3(x) = 6x^3 + 4x^2 + 5$ is the polynomial function of degree $n = 3$	$y_p = e^{(0)x}Q_3 = Ax^3 + Bx^2 + Cx + D$	Nil
(ii) Mixture of Polynomial & Exponential Function in multiplication, e.g. $r(x) = 6xe^{-3x}$	$r(x) = e^{-3x}P_1(x)$ where $\alpha = -3$ & $P_1(x) = 6x$	$y_p = e^{(-3)x}Q_1 = e^{(-3)x}(Ax + B)$	Nil
(iii) Mixture of Polynomial & Exponential Function in '+', e.g. $r(x) = e^{-3x} + 6x^3 + 4x^2 + 5$	*For polynomial function*, $r(x) = e^{(0x)}P_3(x)$ where $\alpha = 0$ & $P_3(x) = 6x^3 + 4x^2 + 5$ is the polynomial function of degree $n = 3$ *For exponential function*, $r(x) = e^{-3x}P_0(x)$ where $\alpha = -3$ & $P_n(x) = 1$ with degree $n = 0$	*For polynomial function*, $y_{p,1} = e^{(0)x}Q_3 = Ax^3 + Bx^2 + Cx + D$ *For exponential function*, $y_{p,2} = e^{-3x}Q_O = Ee^{-3x}$ *For mixture of them*, $y_p = y_{p,1} + y_{p,2}$	**Alternative:** Can be solved separately (i.e. obtain $y_{p,1}$ & $y_{p,2}$) and then combine the result (i.e. $y_p = y_{p,1} + y_{p,2}$) This is known as linear superposition (the linearity principle)

Note: $e^{\pm ix} = cosx \pm i(sinx)$; $Q_n(x)$ & $P_n(x)$ are two polynomial functions with same degree.

Note 3: Now, we will discuss the '***Step 3:*** *To check the linear dependency and give treatment to particular solution if needed*'.

The possible particular solution is proposed according to the RHS function, however, further treatment will be needed to obtain a linearly independent solution *by comparing with the complementary solution*. In fact, the proposed particular solution y_p can be separated into 3 cases depending on

(i) The possible particular solution, $y_p = e^{\alpha x} Q_n(x)$ and

(ii) The complementary solution that in the function of roots m_1 & m_2, i.e.

$$y_c = \begin{Bmatrix} c_1 e^{m_1 x} + c_2 e^{m_2 x} & , where\ m_1 \neq m_2\ \ [\textbf{Case 1}] \\ c_1 e^{(m+i\beta)x} + c_2 e^{(m-i\beta)x} & , where\ m_1 \neq m_2\ \ [\textbf{Case 2}] \\ c_1 x e^{mx} + c_2 e^{mx} & , where\ m_1 = m_2\ \ [\textbf{Case 3}] \end{Bmatrix}$$

The proposed particular solution is illustrated below.

	Case 1 $(\alpha \neq m_1 \,\&\, m_2)$	**Case 2** $(\alpha = m_1\ or\ m_2,\ m_1 \neq m_2)$	**Case 3** $(\alpha = m_1 = m_2)$
Definition	Coefficient α is not equal to coefficients m_1 & m_2	Coefficient α is equal to one of the coefficient $e.g. m_1$ and different with another coefficient m_2	Coefficient α is equal to both coefficients m_1 & m_2
Possible complementary solution for homogeneous ODE	$y_c = \begin{Bmatrix} c_1 e^{m_1 x} + c_2 e^{m_2 x} \\ c_1 e^{(m+i\beta)x} + c_2 e^{(m-i\beta)x} \\ c_1 x e^{mx} + c_2 e^{mx} \end{Bmatrix}$	$y_c = \begin{Bmatrix} c_1 e^{m_1 x} + c_2 e^{m_2 x} \\ c_1 e^{(m+i\beta)x} + c_2 e^{(m-i\beta)x} \end{Bmatrix}$	$y_c = c_1 x e^{mx} + c_2 e^{mx}$
Proposed particular solution for non-homogeneous ODE	$y_p = e^{\alpha x} Q_n(x)$	$y_p = x e^{\alpha x} Q_n(x)$	$y_p = x^2 e^{\alpha x} Q_n(x)$
Observation	If $\alpha \neq m_1 \,\&\, m_2$, **No treatment** is needed for $y_p = e^{\alpha x} Q_n(x)$ because y_p has various forms as y_c (i.e. y_p & y_c are linearly independent)	If $\alpha = m_1\ or\ m_2$, $m_1 \neq m_2$, we can't use $y_p = e^{\alpha x} Q_n(x)$ because this form has the similar form as y_c and cause zero RHS function. (Linearly dependent) **Treatment is needed**.	If $\alpha = m_1 = m_2$, we can't use $y_p = e^{\alpha x} Q_n(x)$ or $y_p = x e^{\alpha x} Q_n(x)$ because these forms have the similar form as y_c and cause zero RHS function. (Linearly dependent) **Treatment is needed**.

Hint: *Multiply the independent variable, x or x^2 to the particular solution* if you found the complementary solution has the similar exponential function as the proposed particular solution. This is known as the cure / treatment to the particular solution.

Note 4: ***Step 4 & 5*** are quite straight forward, the general solution of non-homogeneous ODE consists of complementary solution and particular solution (i.e. $y_{total} = y_c + y_p$).

<u>Example 4.2:</u> Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^x$ [*RHS - Pure Exponential Function*]

<table>
<tr><th>Step 1: Homogeneous Part
i.e. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$</th><th>Step 2: Nonhomogeneous Part
i.e. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^x$</th></tr>
<tr><td>Characteristic equation:
$m^2 - 3m + 2 = 0$
$(m-2)(m-1) = 0$
$m_1 = 2 \,\&\, m_2 = 1$

Comment: Real & distinct roots

Complementary solution:
$y_c = c_1e^{2x} + c_2e^x$</td><td>The method of undetermined coefficient:
RHS: $r(x) = e^{\alpha x}P_n(x)$
where $\alpha = 1, n = 0$
Possible particular solution:
$y_p = e^xQ_0(x) = Ae^x$
Since $\alpha \neq m_1$ and $\alpha = m_2$, treatment is necessary:
$y_p = Axe^x$
Comment:
(i) $y_p = Ae^x$ & $y_c = c_1e^{2x} + c_2e^x$ are linearly dependent.
(ii) $y_p = Axe^x$ & $y_c = c_1e^{2x} + c_2e^x$ are linearly independent.

Solve the coefficient for the proposed particular solution:
$y_p = Axe^x$
Differentiate it, we get: $\frac{dy_p}{dx} = Axe^x + Ae^x$
$\frac{d^2y_p}{dx^2} = Axe^x + 2Ae^x$
Substitute to the ODE equation: $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^x$
>> $(Axe^x + 2Ae^x) - 3(Axe^x + 2Ae^x) + 2(Axe^x) = e^x$
>> $-Ae^x = e^x$
Comparing the coefficients,
>> $e^x: A = -1$
The actual particular solution:
$y_p = -xe^x$</td></tr>
<tr><td colspan="2">The ***complete/ general solution*** to $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^x$ is
$y_{total} = y_c + y_p = c_1e^{2x} + c_2e^x - xe^x$</td></tr>
</table>

Example 4.3: Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 4sin2x$ [*RHS - Pure Sine Function*]

<table>
<tr><th>Step 1: Homogeneous Part
i.e. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$</th><th>Step 2: Nonhomogeneous Part
i.e. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 4sin2x$</th></tr>
<tr><td>
Characteristic equation:
$m^2 - 5m + 6 = 0$
$(m-2)(m-3) = 0$
$m_1 = 2 \;\&\; m_2 = 3$

Comment: Real & distinct roots

Complementary solution:
$y_c = c_1e^{2x} + c_2e^{3x}$
</td><td>
The method of undetermined coefficient:
RHS: $r(x) = 4sin2x$
From Euler's formula: $e^{(2ix)} = \cos(2x) + isin(2x)$
Thus, $Im[e^{(2ix)}] = sin(2x)$
RHS: $r(x) = e^{\alpha x}P_n(x) = 4Im[e^{(2ix)}]$
where $\alpha = 2i, n = 0$

Possible particular solution:
$y_p = e^{2ix}Q_0(x) = Ae^{2ix}$
Since $\alpha \neq m_1$ and m_2, treatment is not needed.
$y_p = Ae^{2ix}$
$y_{p,actual} = Im(y_p)$

Comment:
(i) $y_p = Ae^{2ix}$ & $y_c = c_1e^{2x} + c_2e^{3x}$ are linearly independent.
(ii) <u>Common practice use the y_p in the calculation instead of $Im(y_p)$ for the ease of calculation</u>. Once the y_p is solved, then we can determine the actual y_p using $y_{p,actual} = Im(y_p)$.

Solve the coefficient for the particular solution:
$y_p = Ae^{2ix}$
Differentiate it, we get: $\frac{dy_p}{dx} = 2iAe^{2ix}$
$\frac{d^2y_p}{dx^2} = -4Ae^{2ix}$

Substitute to the ODE equation: $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 4sin2x$
>> $(-4Ae^{2ix}) - 5(2iAe^{2ix}) + 6(Ae^{2ix}) = 4e^{(2ix)}$
>> $i(-10Ae^{2ix}) + 2(Ae^{2ix}) = 4e^{(2ix)}$
</td></tr>
</table>

	Comparing the coefficients, >> $e^{2ix}: 2A - 10Ai = 4$ >> $A = \frac{4}{2-10i} = \frac{2}{1-5i}$ The particular solution: $y_p = \frac{2}{1-5i} e^{2ix}$ >> $y_p = \frac{2}{1-5i}\left(\frac{1+5i}{1+5i}\right)(cos2x + isin2x)$ >> $y_p = \frac{2(1+5i)}{26}(cos2x + isin2x)$ >> $y_p = \frac{(1+5i)}{13}(cos2x + isin2x)$ >> $y_p = \frac{(cos2x-5sin2x)+i(5cos2x+sin2x)}{13}$ The actual particular solution: $y_{p,actual} = Im(y_p)$ >> $y_{p,actual} = Im\left(\frac{(cos2x-5sin2x)+i(5cos2x+sin2x)}{13}\right) = \frac{(5cos2x+sin2x)}{13}$
The ***complete/ general solution*** to $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 4sin2x$ is $y_{total} = y_c + y_p = c_1e^{2x} + c_2e^{3x} + \frac{(5cos2x+sin2x)}{13}$	

Note: Similar procedure as the case of *RHS - Pure Cosine Function*

Example 4.4: Solve $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=e^{-2x}cosx$ [*RHS - Mixture of Exponential & Cosine Function*]

<table>
<tr><th>Step 1: Homogeneous Part
i.e. $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=0$</th><th>Step 2: Nonhomogeneous Part
$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=e^{-2x}cosx$</th></tr>
<tr><td>

Characteristic equation:

$$m^2+4m+5=0$$

$$(m+2)^2-4+5=0$$

$$m=-2\pm\sqrt{-1}$$

$$m_1=-2+\mathrm{i}\ \&\ m_2=-2-\mathrm{i}$$

Comment: A pair of complex conjugates roots

Complementary solution:

$$y_c=c_1e^{(-2+\mathrm{i})x}+c_2e^{(-2-\mathrm{i})x}$$

Comment:

(i) For your extra info., the complementary solution can be converted to $y_c=e^{-2x}(Acosx+Bsinx)$

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The method of undetermined coefficient:

$$\text{RHS: } r(x)=e^{-2x}cosx$$

From Euler's formula: $e^{(\mathrm{ix})}=\cos(x)+isin(x)$

Thus, $Re[e^{(\mathrm{ix})}]=cos(x)$

$$\text{RHS: } r(x)=e^{\alpha x}P_n(x)=e^{-2x}Re[e^{(\mathrm{ix})}]=Re[e^{(-2+\mathrm{i})x}]$$

$$\text{where } \alpha=-2+i, n=0$$

Possible particular solution:

$$y_p=e^{(-2+i)x}Q_0(x)=Ae^{(-2+i)x}$$

Since $\alpha=m_1$ and $\alpha\neq m_2$, treatment is needed.

$$y_p=Axe^{(-2+i)x}$$

$$y_{p,actual}=Re(y_p)$$

Comment:

(i) $y_p=Ae^{(-2+i)x}$ and $y_c=c_1e^{(-2+\mathrm{i})x}+c_2e^{(-2-\mathrm{i})x}$ are linearly dependent

(ii) $y_p=Axe^{(-2+i)x}$ and $y_c=c_1e^{(-2+\mathrm{i})x}+c_2e^{(-2-\mathrm{i})x}$ are linearly independent

(iii) Common practice use the y_p in the calculation instead of $Re(y_p)$ for the ease of calculation. Once the y_p is solved, then we can solve the actual y_p using $y_{p,actual}=Re(y_p)$.

Solve the coefficient for the particular solution:

$$y_p=Axe^{(-2+i)x}$$

Differentiate: $\frac{dy_p}{dx}=(-2+i)Axe^{(-2+i)x}+Ae^{(-2+i)x}$

$$\frac{d^2y_p}{dx^2}=(3-4i)Axe^{(-2+i)x}+(-4+2i)Ae^{(-2+i)x}$$

Substitute to the ODE equation: $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=e^{-2x}cosx$

>> $\left((3-4i)Axe^{(-2+i)x}+(-4+2i)Ae^{(-2+i)x}\right)+4\left((-2+i)Axe^{(-2+i)x}+Ae^{(-2+i)x}\right)+5\left(Axe^{(-2+i)x}\right)=e^{(-2+\mathrm{i})x}$

>> $2iAe^{(-2+i)x}=e^{(-2+\mathrm{i})x}$

</td></tr>
</table>

	Comparing the coefficients, >> $e^{(-2+i)x}: 2iA = 1$ >> $A = \frac{1}{2i}$ The particular solution: $y_p = Axe^{(-2+i)x} = \frac{1}{2i}xe^{(-2+i)x}$ >> $y_p = \frac{1}{2i}\frac{i}{i}xe^{(-2)x}e^{(i)x}$ >> $y_p = -\frac{i}{2}xe^{-2x}(cosx + isinx)$ >> $y_p = -\frac{1}{2}xe^{-2x}(icosx - sinx)$ The actual particular solution: $y_{p,actual} = Re(y_p)$ >> $y_{p,actual} = Re\left(-\frac{1}{2}xe^{-2x}(icosx - sinx)\right) = \frac{1}{2}xe^{-2x}(sinx)$
The ***complete / general solution*** to $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = e^{-2x}cosx$ is $y_{total} = y_c + y_p = c_1e^{(-2+i)x} + c_2e^{(-2-i)x} + \frac{1}{2}xe^{-2x}(sinx)$	

There is an alternative to solve 2nd order non-homogeneous linear ODE problem with RHS sine and cosine functions by using $y_p = Ccosx + Dsinx$. Students are allowed to use either way to solve.

Alternative method to solve the same example: Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = e^{-2x}cosx$

Step 1: Homogeneous Part **i.e.** $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$	**Step 2: Nonhomogeneous Part** $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = e^{-2x}cosx$
Characteristic equation: $m^2 + 4m + 5 = 0$ $(m+2)^2 - 4 + 5 = 0$ $m = -2 \pm \sqrt{-1}$ $m_1 = -2 + \mathrm{i}$ & $m_2 = -2 - \mathrm{i}$ ***Comment***: A pair of complex conjugates roots	The method of undetermined coefficient: RHS: $r(x) = e^{-2x}cosx$ Possible particular solution: $y_p = e^{-2x}(Ccosx + Dsinx)$ Since $y_c = e^{-2x}(Acosx + Bsinx)$ & $y_p = e^{-2x}(Ccosx + Dsinx)$ are linearly dependent, treatment is needed. Actual particular solution: $y_p = xe^{-2x}(Ccosx + Dsinx)$ Solve the coefficient for the particular solution: $y_p = xe^{-2x}(Ccosx + Dsinx)$

<table>
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Complementary solution
[Option 1]
:$y_c = c_1 e^{(-2+\mathrm{i})x} + c_2 e^{(-2-\mathrm{i})x}$

Complementary solution
[Option 2]
$y_c = e^{-2x}(Acosx + Bsinx)$
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<td>
Differentiate it, we get:
$\frac{dy_p}{dx} = e^{-2x}(Ccosx + Dsinx) + x(-2e^{-2x})(Ccosx + Dsinx)$
$+xe^{-2x}(-Csinx + Dcosx)$
$= e^{-2x}(Ccosx + Dsinx)$
$+xe^{-2x}\big((-2C + D)cosx + (-2D + C)sinx\big)$

$\frac{d^2y_p}{dx^2} = (-2e^{-2x})(Ccosx + Dsinx) + e^{-2x}(-Csinx + Dcosx)$
$+e^{-2x}\big((-2C + D)cosx + (-2D + C)sinx\big)$
$+x(-2e^{-2x})\big((-2C + D)cosx + (-2D + C)sinx\big)$
$+xe^{-2x}\big((-2C + D)(-sinx) + (-2D + C)cosx\big)$
$= e^{-2x}\big((-4C + 2D)cosx + (-4D - 2C)sinx\big)$
$+xe^{-2x}\big((3C - 4D)cosx + (3D + 4C)sinx\big)$

Substitute to the ODE equation: $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = e^{-2x}cosx$
>> $\Big(e^{-2x}\big((-4C + 2D)cosx + (-4D - 2C)sinx\big) + xe^{-2x}\big((3C - 4D)cosx + (3D + 4C)sinx\big)\Big) + 4\Big(e^{-2x}(Ccosx + Dsinx) + xe^{-2x}\big((-2C + D)cosx + (-2D + C)sinx\big)\Big) + 5\big(xe^{-2x}(Ccosx + Dsinx)\big) = e^{-2x}cosx$
>> $e^{-2x}\big((2D)cosx + (-2C)sinx\big) = e^{-2x}cosx$

Comparing the coefficients,
>>$e^{-2x}cosx$: $2D = 1$
$D = \frac{1}{2}$
>> $e^{-2x}sinx$: $-2C = 0$
$C = 0$
The particular solution:
$y_p = xe^{-2x}\left(0cosx + \frac{1}{2}sinx\right) = \frac{1}{2}xe^{-2x}sinx$
</td>
</tr>
<tr>
<td colspan="2">
The complete / general solution to $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = e^{-2x}cosx$ is
$y_{total} = y_c + y_p = e^{-2x}(Acosx + Bsinx) + \frac{1}{2}xe^{-2x}(sinx)$
[Comment: Same answer as previous example.]
</td>
</tr>
</table>

Example 4.5: Solve $\frac{d^2y}{dx^2} + 4y = 8x^2$ [*RHS - Pure Polynomial Function*]

Step 1: Homogeneous Part **i.e.** $\frac{d^2y}{dx^2} + 4y = 0$	**Step 2: Nonhomogeneous Part** $\frac{d^2y}{dx^2} + 4y = 8x^2$
Characteristic equation: $m^2 + 4 = 0$ $m = \pm\sqrt{-4}$ $m_1 = 2\text{i}$ & $m_2 = -2\text{i}$ ***Comment***: A pair of complex conjugates roots Complementary solution: $y_c = c_1 e^{(2\text{i})x} + c_2 e^{(-2\text{i})x}$	The method of undetermined coefficient: RHS: $r(x) = e^{\alpha x} P_n(x) = 8x^2$ where $\alpha = 0, n = 2$ Possible particular solution: $y_p = e^{(0)x} Q_2(x) = A_2x^2 + A_1x + A_0$ Since $\alpha \neq m_1$ & m_2, treatment is not needed. ***Comment***: (i) $y_p = A_2x^2 + A_1x + A_0$ and $y_c = c_1 e^{(2\text{i})x} + c_2 e^{(-2\text{i})x}$ are linearly independent Solve the coefficient for the particular solution: $y_p = A_2x^2 + A_1x + A_0$ Differentiate: $\frac{dy_p}{dx} = 2A_2x + A_1$ $\frac{d^2y_p}{dx^2} = 2A_2$ Substitute to the ODE equation: $\frac{d^2y}{dx^2} + 4y = 8x^2$ >> $2A_2 + 4(A_2x^2 + A_1x + A_0) = 8x^2$ >> $4A_2x^2 + 4A_1x + 4A_0 + 2A_2 = 8x^2$ Comparing the coefficients, >> x^2: $A_2 = 2$ >> x: $A_1 = 0$ >> x^0: $A_0 = -\frac{A_2}{2} = -1$ The actual particular solution: $y_p = 2x^2 - 1$

The ***complete / general solution*** to $\frac{d^2y}{dx^2} + 4y = 8x^2$ is

$$y_{total} = y_c + y_p = c_1 e^{(2\text{i})x} + c_2 e^{(-2\text{i})x} + 2x^2 - 1$$

Example 4.6: Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 12y = xe^{4x}$ [*Mixture of Polynomial & Exponential Function in 'x'*]

<table>
<tr>
<th>Step 1: Homogeneous Part
i.e. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 12y = 0$</th>
<th>Step 2: Nonhomogeneous Part
i.e. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 12y = xe^{4x}$</th>
</tr>
<tr>
<td>
Characteristic equation:
$$m^2 - 4m - 12 = 0$$
$$(m-6)(m+2) = 0$$
$$m_1 = 6 \ \& \ m_2 = -2$$

Comment: Real and distinct roots

Complementary solution:
$$y_c = c_1e^{6x} + c_2e^{-2x}$$
</td>
<td>
The method of undetermined coefficient:
$$\text{RHS: } r(x) = e^{\alpha x}P_n(x) = xe^{4x}$$
$$\text{where } \alpha = 4, n = 1$$
Possible particular solution:
$$y_p = e^{(4)x}Q_1(x) = (Ax+B)e^{4x}$$
Since $\alpha \neq m_1 \ \& \ m_2$, treatment is not needed.

Comment:
(i) $y_p = (Ax+B)e^{4x}$ and $y_c = c_1e^{6x} + c_2e^{-2x}$ are linearly independent
Solve the coefficient for the particular solution:
$$y_p = (Ax+B)e^{4x}$$
Differentiate it, we get:
$$\frac{dy_p}{dx} = 4(Ax+B)e^{4x} + (A)e^{4x}$$
$$\frac{d^2y_p}{dx^2} = 16(Ax+B)e^{4x} + 8(A)e^{4x}$$
Substitute to the ODE equation: $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 12y = xe^{4x}$
>> $(16(Ax+B)e^{4x} + 8(A)e^{4x}) - 4(4(Ax+B)e^{4x} + (A)e^{4x}) - 12(Ax+B)e^{4x} = xe^{4x}$
>> $\left((4A-12B)e^{4x}\right) - 12Axe^{4x} = xe^{4x}$
Comparing the coefficients,
>> $xe^{4x}: A = \frac{1}{-12}$
>> $e^{4x}: B = \frac{4A}{12} = -\frac{1}{36}$
The actual particular solution:
$$y_p = \left(\frac{1}{-12}x - \frac{1}{36}\right)e^{4x}$$
</td>
</tr>
<tr>
<td colspan="2">
The complete/ general solution to $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 12y = xe^{4x}$ is
$$y_{total} = y_c + y_p = c_1e^{6x} + c_2e^{-2x} + \left(\frac{1}{-12}x - \frac{1}{36}\right)e^{4x}$$
</td>
</tr>
</table>

Example 4.7: Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 3e^{2x} + 4x$ [*RHS - Mixture of Polynomial & Exponential Function in '+'*]

<table>
<tr><th>Step 1: Homogeneous Part
i.e. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$</th><th>Step 2: Nonhomogeneous Part
i.e. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 3e^{2x} + 4x$</th></tr>
<tr><td>Characteristic equation:
$m^2 - 4m + 3 = 0$
$(m-1)(m-3) = 0$
$m_1 = 1$ & $m_2 = 3$

Comment: Real and distinct roots

Complementary solution:
$y_c = c_1e^{x} + c_2e^{3x}$</td><td>The method of undetermined coefficient:
RHS (1): $r_1(x) = e^{\alpha x}P_n(x) = 3e^{2x}$
where $\alpha = 2, n = 0$
RHS (2): $r_2(x) = e^{\alpha x}P_n(x) = 4x$
where $\alpha = 0, n = 1$

Possible particular solution:
$y_{p,1} = e^{(2)x}Q_0(x) = Ae^{2x}$
$y_{p,2} = e^{(0)x}Q_1(x) = Bx + C$
$y_{p,total} = y_{p,1} + y_{p,2} = Ae^{2x} + Bx + C$
Since $\alpha \neq m_1$ & m_2, treatment is not needed.

Comment:
(i) $y_p = Ae^{2x} + Bx + C$ and $y_c = c_1e^{x} + c_2e^{3x}$ are linearly independent
Solve the coefficient for the particular solution:
$y_p = Ae^{2x} + Bx + C$

Differentiate: $\frac{dy_p}{dx} = 2Ae^{2x} + B$
$\frac{d^2y_p}{dx^2} = 4Ae^{2x}$

Substitute to the ODE equation: $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 3e^{2x} + 4x$
>> $(4Ae^{2x}) - 4(2Ae^{2x} + B) + 3(Ae^{2x} + Bx + C) = 3e^{2x} + 4x$
>> $(-Ae^{2x}) + (3Bx) + 3C - 4B = 3e^{2x} + 4x$</td></tr>
</table>

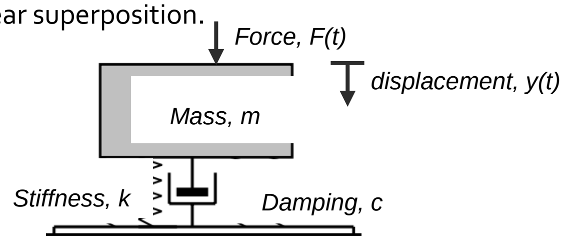
<table>
<tr><td></td><td>Comparing the coefficients,
>> $e^{2x}: A = -3$
>> $x: B = \frac{4}{3}$
>> $x^0: C = \frac{4B}{3} = \frac{16}{9}$

The actual particular solution:
$y_p = -3e^{2x} + \frac{4}{3}x + \frac{16}{9}$</td></tr>
<tr><td colspan="2">The complete/ general solution to $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 3e^{2x} + 4x$ is
$y_{total} = y_c + y_p = c_1e^x + c_2e^{3x} - 3e^{2x} + \frac{4}{3}x + \frac{16}{9}$</td></tr>
</table>

Hint: The example above illustrates the linearity or superposition principle, where the solutions can be added directly as illustrated below.

>> $y_{p,1} = -3e^{2x}$ is the particular solution to $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 3e^{2x}$;
>> $y_{p,2} = \frac{4}{3}x + \frac{16}{9}$ is the particular solution to $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 4x$;
>> $y_{p,total} = y_{p,1} + y_{p,2}$ is the total particular solution to $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 3e^{2x} + 4x$.

Graphical representation of the linear superposition.



$F_1(t) = 3e^{2t}$ → System modeling #1: $m\ddot{y} + c\dot{y} + ky = F_1(t)$ → $y_{p,1} = -3e^{2t}$

$F_2(t) = 4t$ → System modeling #2: $m\ddot{y} + c\dot{y} + ky = F_2(t)$ → $y_{p,2} = \frac{4}{3}t + \frac{16}{9}$

$F_{total}(t) = 3e^{2t} + 4t$ (2 forces acting simultaneously) → System modeling #3: $m\ddot{y} + c\dot{y} + ky = F_1(t) + F_2(t)$ → $y_{p,total} = -3e^{2t} + \frac{4}{3}t + \frac{16}{9}$ (Responses can be obtained by linear superposition)

Recommendation: In engineering application, learning mathematics tool such as the strategy / method to solve ODE problems should not be the main focus, as this can be done perfectly with the use of computer & algorithm. However, it is important for engineers to *evaluate and justify the appropriateness of the selected mathematical tool in solving certain engineering problem*. Furthermore, engineers should learn to *interpret the result and data analysis*. For example, we can obtain the solutions of charge and current from the 2nd order ODE that represents RLC circuit problem. We should plot it and do further analysis to know the characteristic and behaviour of the system. This enables us to *design things / systems in scientific approach (i.e. analytical study in this case) instead of trial and error*.

4.3 SOLVING GENERAL NON-HOMOGENEOUS LINEAR ODE WITH METHOD OF VARIATION OF PARAMETERS

The method of undetermined coefficients discussed previously is *only applicable for simple functions in its RHS, such as a mixture of exponential and polynomial functions*. However, it is impractical to solve complicated function other than exponential and polynomial functions.

To solve 2nd order ODE with *complicated function such as tangent function, Mixture of Polynomial & Exponential Function in '÷', logarithmic function, etc*, we can use the **method of variation of parameters**.

The steps of implementing the **method of variation of parameters** are as follow:

(1) Standard form: $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$

(2) Compute the complementary solution, $y_c = c_1 y_1(x) + c_2 y_2(x)$ by using known method as before, where c_1 & c_2 are arbitrary constants.

(3) Compute the particular solution, $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$
where function $u_1(x) = -\int \frac{r(x)y_2(x)}{W(y_1,y_2)}dx$ and $u_2(x) = \int \frac{r(x)y_1(x)}{W(y_1,y_2)}dx$;

Wronskian, $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{vmatrix}$

However, the drawback of this technique is that it is *time consuming to complete the integration operation* and there are cases where the integration function cannot be solved analytically (using calculus). In this case, we may need an advance tool such as numerical method to solve the integration problem.

Example 4.8: Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{-x} lnx$

Step 1: Homogeneous Part **i.e.** $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$**,**	**Step 2: Nonhomogeneous Part** **i.e.** $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{-x} lnx$
Characteristic equation: $m^2 + 2m + 1 = 0$ $(m+1)^2 = 0$ $m_1 = m_2 = -1$ ***Comment***: Repeated real root Complementary solution: $y_c = c_1 e^{-x} + c_2 x e^{-x}$ ***Comment:*** (i) $y_{c,1} = e^{-x}$ and $y_{c,2} = e^{-x}$ are linearly dependent. (ii) Treatment is done so that $y_{c,1} = e^{-x}$ and $y_{c,2} = xe^{-x}$ are linearly independent. Compute the Wronskian, $W(y_1, y_2)$ $= \begin{vmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{vmatrix}$ $= \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & -xe^{-x} + e^{-x} \end{vmatrix}$ $= e^{-x}(-xe^{-x} + e^{-x}) - (xe^{-x})(-e^{-x})$ $= e^{-2x}$	The method of undetermined coefficient: RHS : $r(x) = e^{\alpha x} P_n(x) = e^{-x} lnx$ where $\alpha = -1, n =$ *not applicable* ***Comment:*** The RHS function is a *complicated function that can't solved by the method of undetermined coefficient*. The method of variation of parameters: $u_1(x) = -\int \frac{r(x) y_2(x)}{W(y_1, y_2)} dx$ $= -\int \frac{(e^{-x} lnx)(xe^{-x})}{e^{-2x}} dx$ $= -\int x lnx\, dx$ Using integration by part method: Let $u = lnx;\ dv = xdx$ $du = \frac{1}{x} dx;\ v = \frac{x^2}{2}$ $u_1(x) = -\int x lnx\, dx = -(uv - \int v du)$ $= -\left(lnx \left(\frac{x^2}{2}\right) - \int \frac{x^2}{2} \frac{1}{x} dx \right)$ $= -\left(lnx \left(\frac{x^2}{2}\right) - \int \frac{x}{2} dx \right)$ $= \left(-lnx \left(\frac{x^2}{2}\right) + \frac{x^2}{4} \right)$ $u_2(x) = \int \frac{r(x) y_1(x)}{W(y_1, y_2)} dx$ $= \int \frac{(e^{-x} lnx)(e^{-x})}{e^{-2x}} dx$ $= \int lnx\, dx$ Using integration by part method: Let $u = lnx;\ dv = dx$, then $du = \frac{1}{x} dx;\ v = x$ $u_2(x) = \int lnx\, dx = (uv - \int v du)$ $= \left(lnx(x) - \int x \frac{1}{x} dx \right)$ $= (lnx(x) - \int 1 dx)$

	$= (lnx(x) - x)$ The particular solution: $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$ $= \left(-lnx\left(\frac{x^2}{2}\right) + \frac{x^2}{4}\right)e^{-x} + (lnx(x) - x)(xe^{-x})$ $= \left(lnx\left(\frac{x^2}{2}\right) - \frac{3x^2}{4}\right)e^{-x}$
The ***complete / general solution*** to $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{-x}lnx$ is $y_{total} = y_c + y_p = c_1e^{-x} + c_2xe^{-x} + \left(lnx\left(\frac{x^2}{2}\right) - \frac{3x^2}{4}\right)e^{-x}$	

Example 4.9: Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x^2+1}$

Step 1: Homogeneous Part i.e. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$	**Step 2: Nonhomogeneous Part** i.e. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x^2+1}$
Characteristic equation: $$m^2 - 2m + 1 = 0$$ $$(m-1)^2 = 0$$ $$m_1 = m_2 = 1$$ ***Comment***: Repeated real root Complementary solution: $$y_c = c_1e^x + c_2xe^x$$ Comment: (i) $y_{c,1} = e^x$ and $y_{c,2} = e^x$ are linearly dependent. (ii) Treatment is done so that $y_{c,1} = e^x$ and $y_{c,2} = xe^x$ are linearly independent. Compute the Wronskian, $W(y_1, y_2)$ $= \begin{vmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{vmatrix}$ $= \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix}$ $= e^x(xe^x + e^x) - (xe^x)(e^x)$ $= e^{2x}$	The method of undetermined coefficient: $$\text{RHS} : r(x) = e^{\alpha x}P_n(x) = \frac{e^x}{x^2+1}$$ $$\text{where } \alpha = 1, n = not\ applicable$$ ***Comment***: The RHS function is a complicated function that can't solved by the method of undetermined coefficient. The method of variation of parameters: $u_1(x) = -\int \frac{r(x)y_2(x)}{W(y_1,y_2)}dx$ $= -\int \frac{\left(\frac{e^x}{x^2+1}\right)(xe^x)}{e^{2x}}dx$ $= -\int \frac{x}{x^2+1}dx$ Using substitution method: Let $u = x^2 + 1$; $du = 2xdx$ $u_1(x) = -\int \frac{1}{u}\frac{du}{2}$ $= -\frac{1}{2}ln\|u\|$ $= -\frac{1}{2}ln\|x^2+1\|$ $u_2(x) = \int \frac{r(x)y_1(x)}{W(y_1,y_2)}dx$ $= \int \frac{\left(\frac{e^x}{x^2+1}\right)(e^x)}{e^{2x}}dx$ $= \int \frac{1}{x^2+1}dx$ Using trigonometric substitution: Let $x = tan\theta, \frac{dx}{d\theta} = sec^2\theta$ $u_2(x) = \int \frac{1}{tan^2\theta+1}sec^2\theta d\theta$ Using $trigonometric$ identity: $tan^2\theta + 1 = sec^2\theta$ $u_2(x) = \int 1\, d\theta = \theta$ $= tan^{-1}x$

	The particular solution: $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$ $= \left(-\frac{1}{2}ln\|x^2+1\|\right)e^x + (tan^{-1}x)(xe^x)$ $= \left(xtan^{-1}x - \frac{1}{2}ln\|x^2+1\|\right)e^x$

The ***complete / general solution*** to $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x^2+1}$ is

$$y_{total} = y_c + y_p = c_1e^x + c_2xe^x + \left(xtan^{-1}x - \frac{1}{2}ln|x^2+1|\right)e^x$$