POWER SERIES SOLUTIONS FOR DIFFERENTIAL EQUATIONS

WEEK 6: POWER SERIES SOLUTIONS FOR DIFFERENTIAL EQUATIONS

6.1 Power series method

Power Series Method

The power series method is the standard basic method for solving linear differential equations with **variable** coefficients. It gives solutions in the form of power series.

Power Series

A power series is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots \quad (1)$$

where a_0, a_1, a_2, … are real constants, called the coefficients of the series, x_0 is a constant, called the center of the series, and x is a variable.

In particular, if $x_0 = 0$, a power series in powers of x is obtained

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Familiar examples of power series:

(i) $$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots$$

(ii) $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

(iii) $$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \pm \cdots$$

(iv) $$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} \pm \cdots$$

(v) $$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

6.1.1 Basic concepts of power series

The *n*th partial sum of (1) is

$$s_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n \tag{2}$$

where n = 0, 1, … . If the terms of s_n are from (1), the remaining expression is

$$R_n(x) = a_{n+1}(x - x_0)^{n+1} + a_{n+2}(x - x_0)^{n+2} + \cdots$$

and is called the remainder of (1) after the term $a_n(x - x_0)^n$.

Example:

For the geometric series

$$1 + x + x^2 + \cdots + x^n + \cdots$$

Then:

$s_1 = 1 + x$ $R_1 = x^2 + x^3 + x^4 + \cdots$

$s_2 = 1 + x + x^2$ $R_2 = x^2 + x^3 + x^4 + \cdots$

etc.

If for some $x = x_1$, $s_n(x)$ converges, that is, $\lim_{n\to\infty} s_n(x_1) = s(x_1)$ then the series (1) **converges**, or is called **convergent** at $x = x_1$; and the number $s(x_1)$ is called the value or sum of (1) at x_1, and can be written as

$$s(x_1) = \sum_{n=0}^{\infty} a_n\,(x_1 - x_0)^n$$

If the sequence is divergent at $x = x_1$, then the series (1) is said to **diverge**, or to be **divergent** at $x = x_1$.

Note:

1. The series (1) converges at $x = x_0$ when all its terms except for the first ao are zero. In unusual cases this may be the only x for which (1) converges.

2. If there are further values of x for which the series (1) converges, these value form an interval, called the **convergence interval**. If this interval is finite, it has the midpoint x_0 so that it is of the form

$$|x - x_0| < R$$

and the series (1) converges for all x such that $|x - x_0| < R$ and diverges for all x such that $|x - x_0| > R$. The number R is called the **radius of convergence** of (1). It can be obtained from either of the following formulas:

$$(a)\quad R = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{|a_m|}} \qquad\qquad (b)\quad R = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \qquad (3)$$

provided these limits exist and are not zero. [If they are infinite, then (1) converges only at the center x_0.]

3. The convergence interval may sometimes be infinite, that is, (1) converges for all x. For example, if the limit in (3a) and (3b) is zero. Then $R = \infty$, for convenience.

4. Since power series are functions of *x* and we know that not every series will in fact exist, it then makes sense to ask if a power series will exist for all *x*. This question is answered by looking at the convergence of the power series. We say that a power series **converges** for *x* = *c* if the series,

$$\sum_{n=0}^{\infty} a_n\,(c - x_0)^n$$

converges. Recall that this series will converge if the limit of partial sums,

$$\lim_{N\to\infty}\sum_{n=0}^{N} a_n\,(c - x_0)^n$$

exists and is finite. In other words, a power series will converge for *x* = *c* if

$$\sum_{n=0}^{\infty} a_n\,(c - x_0)^n$$

is a finite number.

5. A power series will always converge if $x = x_o$. In this case the power series will become

$$\sum_{n=0}^{\infty} a_n\,(c - x_0)^n = a_0$$

With this it is known now that power series are guaranteed to exist for at least one value of *x*. The following fact about the convergence of a power series is derived.

Fact

Given a power series, (1), there will exist a number $0 \le \rho \le \infty$ so that the power series will converge for $|x - x_0| < \rho$ and diverge for $|x - x_0| > \rho$. This number is called the **radius of convergence**.

6.1.2 Test for convergence

1. If $\lim_{n\to\infty} u_n = 0$ the series may be convergent; and

   if $\lim_{n\to\infty} u_n \neq 0$ the series is certainly divergent.

2. Comparison test – useful standard series

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \cdots + \frac{1}{n^p} + \cdots$$

   For $p > 1$, the series converges; for $p < 1$, the series diverges.

3. D'Alembert's Ratio Test for positive terms

   Let $u_1 + u_2 + u_3 + u_4 + \ldots + u_n + \ldots$ be a series of positive terms. Find expressions for u_n and u_{n+1}, that is, the n^{th} term and the $(n + 1)^{th}$ term, respectively, and form the ratio

$$\frac{u_{n+1}}{u_n}$$

   Then, find the limiting value for this ratio,

$$\rho = \lim_{n\to\infty} \frac{u_{n+1}}{u_n}$$

   If $\rho < 1$, the series converges;

   $\rho > 1$, the series diverges;

   $\rho = 1$, the series may converge or diverge and the test gives no definite information.

4. For general series:

   (i) if $\sum |u_n|$ converges, $\sum u_n$ is absolutely convergent

   (ii) if $\sum |u_n|$ diverges, but $\sum u_n$ converges, then $\sum u_n$ is conditionally convergent

Example 6.1:

Find the radius of convergence of the following series.

1. $$\sum_{k=1}^{\infty} \frac{k}{2^k}$$

Solution:

$$\rho = \lim_{k\to\infty} \left|\frac{k+1}{2^{k+1}} \cdot \frac{2^k}{k}\right| = \lim_{k\to\infty} \left|\frac{k+1}{2k}\right| = \frac{1}{2} \lim_{k\to\infty} \left|\frac{k+1}{k}\right| = \frac{1}{2} \lim_{k\to\infty} \left|1+\frac{1}{k}\right| = \frac{1}{2}$$

The series converges

2. $$\sum_{k=1}^{\infty} \frac{(-1)^k (x-3)^k}{3^k (k+1)}$$

Solution:

$$\rho = \lim_{k\to\infty} \left|\frac{(-1)^{k+1}(x-3)^{k+1}}{3^{k+1}(k+1+1)} \cdot \frac{3^k (k+1)}{(-1)^k (x-3)^k}\right| = \lim_{k\to\infty} \left|\frac{(-1)(x-3)(k+1)}{3(k+2)}\right|$$

$$= \frac{|(x-3)|}{3} \lim_{k\to\infty} \left|\frac{(-1)(k+1)}{(k+2)}\right| = \frac{|(x-3)|}{3} \cdot 1 = \frac{|(x-3)|}{3}$$

Series converges when $\rho < 1$

$$\frac{|(x-3)|}{3} < 1$$

$$|(x-3)| < 3$$

$$-(x-3) < 3 \quad \Rightarrow \quad x > 0$$

or

$$(x-3) < 3 \quad \Rightarrow \quad x < 6$$

Convergence interval (0, 6)

Radius of convergence 3

6.1.3 Operations of power series

Three permissible operations on power series: differentiation, addition, and multiplication.

(1) Termwise differentiation

A power series may be differentiated term by term. If

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

converges for $|x - x_0| < R$ where $R > 0$, then the series obtained by differentiating term by term also converges for those x and represents the derivative y' of y for those x, that is,

$$y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$$

Similarly,

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}$$

and so on.

(2) Termwise addition

Two power series may be added term by term. If the series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad \text{and} \qquad \sum_{n=0}^{\infty} b_n (x - x_0)^n$$

have positive radii of convergence and their sums are $f(x)$ and $g(x)$, respectively, then the series

$$\sum_{n=0}^{\infty} (a_n + b_n)(x - x_0)^n$$

converges and represent $f(x) + g(x)$ for each x that lies in the interior of the convergence interval of each of the given series.

(3) Termwise multiplication

Two power series may be multiplied term by term. Suppose that

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad \text{and} \qquad \sum_{n=0}^{\infty} b_n (x - x_0)^n$$

have positive radii of convergence and let $f(x)$ and $g(x)$ be their sums, respectively. Then the series obtained by multiplying each term of the first series by each term of the second series and collecting like powers of $x - x_0$, that is,

$$\sum_{n=0}^{\infty}(a_0b_n + a_1b_{n-1} + \cdots + a_nb_0)(x - x_0)^n$$
$$= a_0b_0 + (a_0b_1 + a_1b_0)(x - x_0) + (a_0b_2 + a_1b_1 + a_2b_0)(x - x_0)^2 + \cdots$$

converges and represents *f*(*x*)*g*(x) for each *x* in the interior of convergence interval of each of the given series.

6.1.4 Vanishing all coefficients – a condition that is a basic tool of the power series method

If a power series has a positive radius of convergence and a sum that is identically zero throughout its interval of convergence, then each coefficient of the series is zero.

Sifting summation indices

(1) An index of summation is a dummy and can be changed.

Example:

$$\sum_{n=1}^{\infty}\frac{3^n n^2}{n!} = \sum_{k=1}^{\infty}\frac{3^k k^2}{k!} = 1 + 18 + \frac{81}{2} + \cdots.$$

(2) An index of summation can be "shifted".

If set $n = s + 2$, then $s = n - 2$, and

$$\sum_{n=2}^{\infty} n(n-1)a_n\,(x - x_0)^{n-2} = \sum_{s=0}^{\infty}(s+2)(s+1)a_{s+2}x^s = 2a_2 + 6a_3x + 12a_4x^2 + \cdots$$

When writing the sum of two series,

$$x^2\sum_{n=2}^{\infty} n(n-1)a_n\,x^{n-2} + 2\sum_{n=1}^{\infty} na_n\,x^{n-1}$$
$$= x^2(2a_2 + 6a_3x + 12a_4x^2 + \cdots) + 2(a_1 + 2a_2x + 3a_3x^2 + \cdots$$

as a single series; firstly, take x^2 and 2, respectively, inside the summation, obtaining

$$\sum_{n=2}^{\infty} n(n-1)x^n + \sum_{n=1}^{\infty} 2na_nx^{n-1}$$

and then set $n = s$ and $n - 1 = s$, respectively, obtaining

$$\sum_{s=2}^{\infty} s(s-1)a_s\, x^s + \sum_{s=0}^{\infty} 2(s+1)\, a_{s+1}x^s$$

where $s = 2$ can be replaced by $s = 0$, so that

$$\sum_{s=0}^{\infty}[s(s-1)a_s + 2(s+1)a_{s+1}]\, x^s = 2a_1 + 4a_2x + (2a_2 + 6a_3)x^2 + (6a_3 + 8a_4)x^3 + \cdots$$

Theorem (*Existence of power series solution*)

If the functions *p*, *q*, and *r* in the differential equation

(4) $$y'' + p(x)y' + q(x)y = r(x)$$

are analytic at $x = x_0$, then every solution $y(x)$ of (4) is analytic at $x = x_0$ and can thus be represented by a power series in powers $x - x_0$ with radius of convergence $R > 0$.

<u>Example 6.2:</u>

Find the series of the following functions.

1. e^{x^2}

Solution:

$$e^{x^2} = \sum_{m=0}^{\infty} \frac{(x^2)^m}{m!} = 1 + x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \cdots$$

$$= 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \cdots$$

2. $e^x + \sin x$

Solution:

$$e^x + \sin x = \sum_{m=0}^{\infty} \frac{x^m}{m!} + \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + x - \frac{x^3}{3!} + \frac{x^5}{5!} \pm \cdots$$

$$= 1 + 2x + \frac{x^2}{2} + \frac{x^4}{4!} + \frac{2x^5}{5!} + \cdots$$

3. $e^x(\cos x)$

Solution:

$$e^x(\cos x) = \left(\sum_{m=0}^{\infty} \frac{x^m}{m!}\right)\left(\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!}\right)$$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \pm \cdots\right)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + x - \frac{x^3}{2!} + \frac{x^5}{4!} + \cdots$$

6.1.5 Idea of the power series method

Before finding series solutions to differential equations; we need to determine when we can find series solutions to differential equations with nonconstant coefficients. So, let's start with the differential equation,

$$p(x)y'' + q(x)y' + r(x)y = 0 \tag{5}$$

To this point we've only dealt with constant coefficients. However, with series solutions we can now have nonconstant coefficient differential equations. Also, here we will be dealing only with polynomial coefficients.

Now, we say that $x = x_o$ is an **ordinary point** if provided both

$$\frac{q(x)}{p(x)} \quad \text{and} \quad \frac{r(x)}{p(x)}$$

are analytic at $x = x_o$. That is to say that these two quantities have Taylor series around $x = x_o$. Since, we are only dealing with coefficients that are polynomials so this will be equivalent to saying that

$$p(x_0) \neq 0$$

for most of the problems.

If a point is not an ordinary point we call it a **singular point**.

The basic idea to finding a series solution to a differential equation is to assume that we can write the solution as a power series in the form,

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{6}$$

and then try to determine what the a_n's need to be. We will only be able to do this if the point $x = x_o$, is an ordinary point. We will usually say that (6) is a series solution around $x = x_o$.

Example 6.3:

1. Find a series solution around x_o = 0 for the following differential equation.

$$y'' - xy = 0$$

Solution:

In this case, $p(x) = 1$; hence for this differential equation every point is an ordinary point.

Assume solution:

$$y = \sum_{n=0}^{\infty} a_n (x-0)^n = \sum_{n=0}^{\infty} a_n x^n$$

Then,

$$y' = \sum_{n=1}^{\infty} n\, a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n\,(n-1) a_n x^{n-2}$$

Step1: Plugging into the differential equation

$$\sum_{n=2}^{\infty} n\,(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n = 0$$

Step 2: Get all the coefficients moved into the series.

$$\sum_{n=2}^{\infty} n\,(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Step 3: Shift the first series down by 2 and the second series up by 1 to get both of the series in terms of x^n

$$\sum_{n=0}^{\infty} (n+2)\,(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

Step 4: Get the two series starting at the same value of *n*. The only way to do that for this problem is to strip out the *n* = 0 term

$$(2)(1) a_2 x^0 + \sum_{n=1}^{\infty} (n+2)\,(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} - a_{n-1}] x^n = 0$$

Step 5: Set all the coefficients equal to zero. The *n* = 0 coefficient is in front of the series and the *n* = 1,2,3… are all in the series. So, setting coefficient equal to zero gives,

$n = 0$ $\quad\quad$ $2a_2 = 0$

$n = 1, 2, 3, \ldots$ $\quad\quad$ $(n+2)(n+1)a_{n+2} - a_{n-1} = 0$

Step 6: Solving the first as well as **the recurrence relation** gives

$n = 0$ $\quad\quad$ $a_2 = 0$

$n = 1, 2, 3, \ldots$ $\quad\quad$ $a_{n+2} -= \dfrac{a_{n-1}}{(n+2)(n+1)}$

Step 7: Start plugging in values of *n*

$n = 1$ $\quad\quad$ $a_3 = \dfrac{a_0}{(3)(2)}$

$n = 2$ $\quad\quad$ $a_4 = \dfrac{a_1}{(4)(3)}$

$n = 3$ $\quad\quad$ $a_5 = \dfrac{a_2}{(5)(4)} = 0$

$n = 4$ $\quad\quad$ $a_6 = \dfrac{a_3}{(6)(5)} = \dfrac{a_0}{(6)(5)(3)(2)}$

$n = 5$ $\quad\quad$ $a_7 = \dfrac{a_4}{(7)(6)} = \dfrac{a_1}{(7)(6)(4)(3)}$

$n = 6$ $\quad\quad$ $a_8 = \dfrac{a_5}{(8)(7)} = 0$

$\vdots$

$$a_{3k} = \frac{a_0}{(2)(3)(5)(6)\cdots(3k-1)(3k)} \qquad k = 1, 2, 3, \cdots$$

$$a_{3k+1} = \frac{a_1}{(3)(4)(6)(7)\cdots(3k)(3k+1)} \qquad k = 1, 2, 3, \cdots$$

$$a_{3k+2} = 0 \qquad k = 0, 1, 2, \cdots$$

Note: Every third coefficient is zero. The formulas here are somewhat unpleasant and not all that easy to see the first time around. These formulas will not work for *k = o*.

Step 8: Get the solution

$$\begin{aligned} y(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots + a_{3k} x^{3k} + a_{3k+1} x^{3k+1} + \cdots \\ &= a_0 + a_1 x + \frac{a_0}{6} x^2 + \frac{a_1}{12} x^4 + \cdots + \frac{a_0}{(2)(3)(5)(6)\cdots(3k-1)(3k)} x^{3k} \\ &\quad + \frac{a_1}{(3)(4)(6)(7)\cdots(3k)(3k+1)} x^{3k+1} + \cdots \end{aligned}$$

Step 9: Collect up the terms that contain the same coefficient, factor the coefficient out and write the results as a new series

$$y(x) = a_0\left[1+\sum_{k=1}^{\infty}\frac{x^{3k}}{(2)(3)(5)(6)\cdots(3k-1)(3k)}\right]+a_1\left[x+\sum_{k=1}^{\infty}\frac{x^{3k+1}}{(3)(4)(6)(7)\cdots(3k)(3k+1)}\right]$$

Note: The series could not start at *k* = 0 since the general term doesn't hold for *k* = 0

2. Find the first four terms in each portion of the series solution around x_o = -2 for the following differential equation

$$y'' - xy = 0$$

Solution:

In this case, $p(x) = 1$; hence for this differential equation every point is an ordinary point.

Assume solution:

$$y = \sum_{n=0}^{\infty} a_n\,(x-(-2))^n = \sum_{n=0}^{\infty} a_n\,(x+2)^n$$

Then,

$$y' = \sum_{n=1}^{\infty} n\,a_n(x+2)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n\,(n-1)a_n(x+2)^{n-2}$$

Step 1: Plugging into the differential equation

$$\sum_{n=2}^{\infty} n\,(n-1)a_n(x+2)^{n-2} - x\sum_{n=0}^{\infty} a_n\,(x+2)^n = 0$$

Step 2: Get all the coefficients moved into the series. There is a difference between this example and the previous example. In this case we can't just multiply the *x* into the second series since in order to combine with the series it must be *x* + 2. Therefore we will first need to modify the coefficient of the second series before multiplying it into the series.

$$\sum_{n=2}^{\infty} n\,(n-1)a_n(x+2)^{n-2} - (x+2-2)\sum_{n=0}^{\infty} a_n\,(x+2)^n = 0$$

$$\sum_{n=2}^{\infty} n\,(n-1)a_n(x+2)^{n-2} - (x+2)\sum_{n=0}^{\infty} a_n\,(x+2)^n + 2\sum_{n=0}^{\infty} a_n\,(x+2)^n = 0$$

$$\sum_{n=2}^{\infty} n\,(n-1)a_n(x+2)^{n-2} - \sum_{n=0}^{\infty} a_n\,(x+2)^{n+1} + \sum_{n=0}^{\infty} 2a_n\,(x+2)^n = 0$$

Note: Now have three series to work with.

Step 3: Need to shift the first series down by 2 and the second series up by 1 to get common exponents in all the series

$$\sum_{n=0}^{\infty}(n+2)\,(n+1)a_{n+2}(x+2)^n - \sum_{n=1}^{\infty} a_{n-1}\,(x+2)^n + \sum_{n=0}^{\infty} 2a_n\,(x+2)^n = 0$$

Step 4: Combine the series by stripping out the n = 0 terms from both the first and third series

$$2a_2 + \sum_{n=1}^{\infty}(n+2)\,(n+1)a_{n+2}(x+2)^n - \sum_{n=1}^{\infty} a_{n-1}\,(x+2)^n + 2a_0 + \sum_{n=1}^{\infty} 2a_n\,(x+2)^n = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} - a_{n-1} + 2a_n](x+2)^n = 0$$

Step 5: Set all the coefficients equal to zero.

$n = 0$ $\qquad 2a_2 + 2a_0 = 0$

$n = 1, 2, 3, \dots$ $\qquad (n+2)(n+1)a_{n+2} - a_{n-1} + 2a_n = 0$

Step 6: Solve the first as well as **the recurrence relation**. In the first case there are two options, we can solve for a_2 or we can solve for a_0. Out of habit I'll solve for a_0. In the recurrence relation we'll solve for the term with the largest subscript

$n = 0$ $\qquad a_2 = -a_0$

$n = 1, 2, 3, \dots$ $\qquad a_{n+2} = \dfrac{a_{n-1} - 2a_n}{(n+2)(n+1)}$

Note 1: This example we won't be having every third term drop out as we did in the previous example.

Note 2: At this point we'll also acknowledge that the instructions for this problem are different as well. We aren't going to get a general formula for the a_n's this time so we'll have to be satisfied with just getting the first couple of terms for each portion of the solution. This is often the case for series solutions. Getting general formulas for the a_n's is the exception rather than the rule in these kinds of problems.

Step 7: Start plugging in values of n. To get the first four terms we'll just start plugging in terms until we've got the required number of terms. Note that we will already be starting with an a_0 and an a_1 from the first two terms of the solution so all we will need are three more terms with an a_0 in them and three more terms with an a_1 in them

$n = 0$ $\qquad a_2 = -a_0$

$n = 1$ $\qquad a_3 = \dfrac{a_0 - 2a_1}{(3)(2)} = \dfrac{a_0}{6} - \dfrac{a_1}{3}$

$n = 2$ $$a_4 = \frac{a_1 - 2a_2}{(4)(3)} = \frac{a_1 - 2(-a_0)}{(4)(3)} = \frac{a_0}{6} + \frac{a_1}{12}$$

$n = 3$ $$a_5 = \frac{a_2 - 2a_3}{(5)(4)} = \frac{a_0}{20} - \frac{1}{10}\left(\frac{a_0}{6} - \frac{a_1}{3}\right) = -\frac{a_0}{15} + \frac{a_1}{30}$$

Step 8: Get the solution

$$\begin{aligned} y(x) &= a_0 + a_1(x+2) + a_2(x+2)^2 + a_3(x+2)^3 + a_4(x+2)^4 + a_5(x+2)^5 + \cdots \\ &= a_0 + a_1(x+2) - a_0(x+2)^2 + \left(\frac{a_0}{6} - \frac{a_1}{3}\right)(x+2)^3 + \left(\frac{a_0}{6} + \frac{a_1}{12}\right)(x+2)^4 \\ &\quad + \left(-\frac{a_0}{15} + \frac{a_1}{30}\right)(x+2)^5 + \cdots \end{aligned}$$

Step 9: Collect up the terms that contain the same coefficient, factor the coefficient out and write the results as a new series

$$\begin{aligned} y(x) &= a_0\left\{1 - (x+2)^2 + \frac{1}{6}(x+2)^3 + \frac{1}{6}(x+2)^4 - \frac{1}{15}(x+2)^5 + \cdots\right\} \\ &\quad + a_1\left\{(x+2) - \frac{1}{3}(x+2)^3 + \frac{1}{12}(x+2)^4 + \frac{1}{30}(x+2)^5 + \cdots\right\} \end{aligned}$$

Note: That's the solution for this problem as far as we're concerned. Notice that this solution looks nothing like the solution to the previous example. It's the same differential equation, but changing x_o completely changed the solution.

3. Determine a series solution about $x_o = 0$ for the following initial value problem.

$$y'' - 2xy' + y = 0, \qquad y(0) = 1, \qquad y'(0) = 1$$

Solution:

Assume solution:

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then,

$$y' = \sum_{n=1}^{\infty} n\, a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n\,(n-1) a_n x^{n-2}$$

Step 1: Plugging into the differential equation

$$\sum_{n=2}^{\infty} n\,(n-1) a_n x^{n-2} - 2x \sum_{n=1}^{\infty} n a_n\, x^{n-1} + \sum_{n=0}^{\infty} a_n\, x^n = 0$$

Step 2: Get all the coefficients moved into the series.

$$\sum_{n=2}^{\infty} n\,(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} 2na_n\, x^n + \sum_{n=0}^{\infty} a_n\, x^n = 0$$

Step 3: Need to shift the first series down by 2 to get common exponents in all the series

$$\sum_{n=0}^{\infty} (n+2)\,(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} 2na_n\, x^n + \sum_{n=0}^{\infty} a_n\, x^n = 0$$

Step 4: Combine the series by stripping out the n = 0 terms from both the first and third series

$$2a_2 x^0 + \sum_{n=1}^{\infty} (n+2)\,(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} 2na_n\, x^n + a_0 x^0 + \sum_{n=1}^{\infty} a_n\, x^n = 0$$

$$(a_0 + 2a_2)x^0 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - 2na_n + a_n]\, x^n = 0 = \; 0x^0 + \sum_{n=0}^{\infty} [0]\, x^n$$

Step 5: Set all the coefficients equal to zero

$n = 0$ $\qquad a_0 + 2a_2 = 0$

$n = 1, 2, 3, \ldots$ $\qquad (n+2)(n+1)a_{n+2} - 2na_n + a_n = 0$

Step 6: Solve the first as well as **the recurrence relation**.

$n = 0$ $\qquad a_2 = -\dfrac{a_0}{2}$

$n = 1, 2, 3, \ldots$ $\qquad a_{n+2} = \dfrac{(2n-1)a_n}{(n+2)(n+1)}$

Step 7: Start plugging in values of n.

$n = 0$ $\qquad a_2 = -\dfrac{a_0}{2}$

$n = 1$ $\qquad a_3 = \dfrac{a_1}{6}$

$n = 2$ $\qquad a_4 = \dfrac{3a_2}{12} = \dfrac{a_2}{4} = -\dfrac{a_0}{8}$

$n = 3$ $\qquad a_5 = \dfrac{5a_3}{20} = \dfrac{a_3}{4} = \dfrac{a_1}{24}$

Note: Can choose any arbitrary constants for a_0 and a_1

Step 8: Get the solution

$$y(x) = \sum_{n=0}^{\infty} a_n\, x^n = \; a_0 + \; a_1 x + \; a_2 x^2 + \; a_3 x^3 + \cdots$$

$$= a_0 + a_1 x + \left(-\frac{a_0}{2}\right)x^2 + \frac{a_1}{6}x^3 + \left(-\frac{a_0}{8}\right)x^4 + \frac{a_1}{24}x^5 + \cdots$$

Step 9: Collect up the terms that contain the same coefficient, factor the coefficient out and write the results as a new series

$$y(x) = a_0 \left[1 - \frac{x^2}{2} - \frac{x^4}{8} + \cdots\right] + a_1 \left[x + \frac{x^3}{6} + \frac{x^5}{24} + \cdots\right]$$

Step 10: Applying the initial conditions gives values for a_0 and a_1

$$y(0) = 1 \quad \Rightarrow \quad a_0 = 1$$

$$y'(0) = 1 \quad \Rightarrow \quad a_1 = 1$$

Step 11: Write out the particular solution

$$y(x) = \left[1 - \frac{x^2}{2} - \frac{x^4}{8} + \cdots\right] + \left[x + \frac{x^3}{6} + \frac{x^5}{24} + \cdots\right]$$

Solutions About Singular Points

The power series method for solving linear differential equations with variable coefficients no longer works when solving the differential equation about a singular point. It appears that some features of the solutions of such equations of the most importance for applications are largely determined by their behavior near their singular points. Frobenius method is usually used to solve the differential equation about a regular singular point. This method does not always yield two infinite series solutions. When only one solution is found, a certain formula can be used to get the second solution.

The two differential equations

$$(a) \qquad y'' + xy = 0 \qquad\qquad (b) \qquad xy'' + y = 0 \qquad\qquad (7)$$

are similar only in that they are both examples of simple linear second-order differential equations with variable coefficients. For (7a), x = 0 is an ordinary point; hence, there is no problem in finding two distinct power series solution centered at that point. In contrast, x = 0 is a singular point for (7b), finding two infinite series solutions about that point becomes more difficult task.

For the homogeneous second-order linear differential equation

$$A(x)y'' + B(x)y' + C(x)y = 0 \qquad (8)$$

The singular points are simply points where $A(x)$ = 0 if the functions A, B, and C are polynomials having no common factors.

For example, $x = 0$ is the only singular point of the Bessel equation of order n,

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

whereas the Legendre equation of order n,

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

has two singular points $x = -1$ and $x = 1$.

Note: Usually, only the case in which $x = 0$ is a singular point of Equation (7) is considered. A differential equation having $x = a$ as a singular point is easily transformed by the substitution $t = x - a$ into one having a corresponding singular point at 0.

Types of Singular Points

A differential equation having a singular point at 0 ordinarily will not have power series solutions of the form

$$y(x) = \sum c_n x^n$$

So the straightforward method of power series fails in this case.

A *singular point* x_0 of a linear differential equation

$$A(x)y'' + B(x)y' + C(x)y = 0$$

is further classified as either *regular* or *irregular*. The classification depends on the functions P and Q in the standard form

$$y'' + P(x)y' + Q(x)y = 0$$

Definition (Regular or Irregular Singular Points)

A singular point x_0 is said to be a regular singular point of the differential equation (8) if the functions

$$p(x) = (x - x_0)P(x) \qquad \text{and} \qquad q(x) = (x - x_0)^2 Q(x)$$

are both analytic at x_0. A singular point that is not regular is said to be irregular singular point of the equation.

Quick Visual Check (Regular or Irregular Singular Points)

If $x - x_0$ appears at most to the first power in the denominator of P(x) and at most to the second power in the denominator of Q(x), then $x - x_0$ is a regular singular point.

Example 6.4:

Find the singular point(s) for the differential equation

$$(x^2-4)^2y''+3(x-2)y'+5y=0$$

Answer

Divide the equation with

$$(x^2-4)^2=(x-2)^2(x+2)^2$$

and reduce the coefficients to the lowest terms, produce

$$P(x)=\frac{3}{(x-2)(x+2)^2} \qquad \text{and} \qquad Q(x)=\frac{5}{(x-2)^2(x+2)^2}$$

Test *P*(*x*) and *Q*(*x*)

(i) For x = 2 to be a regular point, the factor x – 2 can appear at most to the first power in the denominator of *P*(*x*) and at most to the second power in the denominator of *Q*(*x*). A check of the denominators of *P*(*x*) and *Q*(*x*) shows that both these conditions are satisfied, so x _ 2 is a regular singular point. Alternatively, the same conclusion is madeby noting that both rational functions

$$p(x)=(x-2)P(x)=\frac{3}{(x+2)^2} \qquad \text{and} \qquad q(x)=(x-2)^2Q(x)=\frac{5}{(x+2)^2}$$

are analytic at x -2.

(ii) Now since the factor x - (-2) = x + 2 appears to the second power in the denominator of *P*(*x*), we can conclude immediately that x = -2 is an irregular singular point of the equation. This also follows from the fact that

$$p(x)=(x+2)P(x)=\frac{3}{(x-2)(x+2)}$$

is not analytic at x = -2.