FROBENIUS METHOD

WEEK 7: FROBENIUS METHOD

7.1 Solutions about singular points

The power series method for solving linear differential equations with variable coefficients no longer works when solving the differential equation about a singular point. It appears that some features of the solutions of such equations of the most importance for applications are largely determined by their behavior near their singular points. Frobenius method is usually used to solve the differential equation about a regular singular point. *This method does not always yield two infinite series solutions. When only one solution is found, a certain formula can be used to get the second solution*.

Reduction of Order

The "reduction of order method" is a method for converting any linear differential equation to another linear differential equation of lower order, and then constructing the general solution to the original differential equation using the general solution to the lower-order equation.

Reduction of Order for Homogeneous Linear Second-Order Equations

This method is for finding a general solution to some homogeneous linear second-order differential equation

$$ay'' + by' + cy = 0$$

where a, b, and c are known functions with $a(x)$ never being zero on the interval of interest. Then assume that there is already one nontrivial particular solution $y_1(x)$ to this generic differential equation.

Now, the details in using the reduction of order method to solve the above:

Step 1: Let

$$y = y_1 u$$

Then, using the product rule, derive y' and y'':

$$y' = (y_1 u)' = y_1' u + y_1 u'$$

and

$$\begin{aligned} y'' = (y')' &= (y_1' u + y_1 u')' \\ &= (y_1' u)' + (y_1 u')' \\ &= (y_1'' u + y_1' u') + (y_1' u' + y_1 u'') \\ &= y_1'' u + 2y_1' u' + y_1 u'' \end{aligned}$$

Step 2: Plug the formulas just computed for *y* , *y′* and *y″* into the differential equation, group together the coefficients for u and each of its derivatives, and simplify as far as possible.

$$
\begin{aligned}
0 &= ay'' + by' + cy \\
&= a[y_1''u + 2y_1'u' + y_1u''] + b[y_1'u + y_1u'] + c[y_1u] \\
&= ay_1''u + 2ay_1'u' + ay_1u'' + by_1'u + by_1u' + cy_1u \\
&= ay_1u'' + [2ay_1' + by_1']u' + [ay_1'' + by_1' + cy_1]u
\end{aligned}
$$

The differential equation becomes

$$Au'' + Bu' + Cu = 0$$

where

$$A = ay_1$$

$$B = 2ay_1' + by_1'$$

$$C = ay_1'' + by_1' + cy_1$$

But remember y_1 is a solution to the homogeneous equation

$$ay'' + by' + cy = 0$$

Consequently,

$$C = ay_1'' + by_1' + cy_1 = 0$$

and the differential equation for *u* automatically reduces to

$$Au'' + Bu' = 0$$

The *u* term always drops out.

Step 3: Now find the general solution to the second-order differential equation just obtained for *u*

$$Au'' + Bu' = 0$$

via the substitution method:

(a) Let $u' = v$.

Thus,

$$u'' = v' = \frac{dv}{dx}$$

To convert the second-order differential equation for *u* to the first-order differential equation for *v*:

$$A\frac{dv}{dx} + Bv = 0$$

Note: This first-order differential equation will be both linear and separable.

(b) Find the general solution *v*(*x*) to this first-order equation.

(c) Using the formula just found for v , integrate the substitution formula $u' = v$ to obtain the formula for u

$$u(x) = \int v(x)dx$$

Don't forget all the arbitrary constants.

Step 4: Finally, plug the formula just obtained for $u(x)$ into the first substitution $y = y_1 u$ used to convert the original differential equation for y to a differential equation for u. The resulting formula for $y(x)$ will be a general solution for that original differential equation.

To illustrate the method, use the differential equation

$$x^2 y'' - 3xy' + 4y = 0$$

Note that the first coefficient, x^2 , vanishes when $x = 0$. So $x = 0$ ought not be in any interval of interest for this equation and solution should be found over the intervals (0, ∞) and (−∞, 0). Before starting the reduction of order method, one nontrivial solution y_1 is needed to the differential equation. Ways for finding that first solution will be discussed in later chapters. For now let us just observe that if

$$y_1(x) = x^2$$

then

$$\begin{aligned} x^2 y_1'' - 3xy_1' + 4y_1 &= x^2 \frac{d^2}{dx^2}[x^2] - 3x \frac{d}{dx}[x^2] + 4[x^2] \\ &= x^2[2 \cdot 1] - 3x[2x] + 4x^2 \\ &= x^2[2 - (3 \cdot 2) + 4] = 0 \end{aligned}$$

Thus, one solution to the above differential equation is $y_1(x) = x^2$

Step 1:

$$y = y_1 u = x^2 u$$

The derivatives of y are:

$$y' = (x^2 u)' = 2xu + x^2 u'$$

and

$$\begin{aligned} y'' = (y')' &= (2xu + x^2 u')' \\ &= (2xu)' + (x^2 u')' \\ &= (2u + 2xu') + (2xu' + x^2 u'') \\ &= 2u + 4xu' + x^2 u'' \end{aligned}$$

Step 2:

$$0 = x^2 y'' - 3xy' + 4y$$

$$= x^2[2u + 4xu' + x^2u''] - 3x[2xu + x^2u'] + 4[x^2u]$$
$$= 2x^2u + 4x^3u' + x^4u'' - 6x^2u - 3x^3u' + 4x^2u$$
$$= x^4u'' + [4x^3 - 3x^3]u' + [2x^2 - 6x^2 + 4x^2]u$$
$$= x^4u'' + x^3u' + 0 \cdot u$$

So, the resulting differential equation for *u* is

$$x^4u'' + x^3u' = 0$$

Further simplify by dividing x^4

$$u'' + \frac{1}{x}u' = 0$$

Step 3: Let $v = u'$ and $v' = u''$. Hence, the above differential equation becomes

$$\frac{dv}{dx} + \frac{1}{x}v = 0$$

Equivalently,

$$\frac{dv}{dx} = -\frac{1}{x}v$$

This is separable first-order differential equation.

$$\frac{1}{v}\frac{dv}{dx} = -\frac{1}{x}$$
$$\int \frac{1}{v}dv = \int -\frac{1}{x}dx$$
$$ln|v| = -ln|x| + C_0$$
$$v = \pm e^{-ln|x| + C_0}$$
$$v = \pm x^{-1}e^{C_0} = \frac{C_1}{x}$$

Since $u' = v$, then

$$u(x) = \int v(x)dx = \int \frac{C_1}{x}dx = C_1 ln|x| + C_2$$

Step 4: Here, $y_1(x) = x^2$

$$y = y_1u = x^2[C_1 ln|x| + C_2]$$
$$= C_1x^2 ln|x| + C_2x^2$$

This is the general solution to the differential equation $x^2y'' - 3xy' + 4y = 0$. The general solution obtained can be viewed as a linear combination of the two functions

$$y_1(x) = x^2 \qquad \text{and} \qquad y_2 = x^2 ln|x|$$

Since the C_1 and C_2 in the above formula for $y(x)$ are arbitrary constants, and y_2 is given by that formula for y with $C_1 = 1$ and $C_2 = 0$, it must be that this y_2 is another particular solution to our original homogeneous linear differential equation. What's more, it is clearly not a constant multiple of y_1.

The two differential equations

$$(a) \qquad y'' + xy = 0 \qquad\qquad (b) \qquad xy'' + y = 0 \qquad\qquad (7)$$

are similar only in that they are both examples of simple linear second-order differential equations with variable coefficients. For (7a), $x = 0$ is an ordinary point; hence, there is no problem in finding two distinct power series solution centered at that point. In contrast, $x = 0$ is a singular point for (7b), finding two infinite series solutions about that point becomes more difficult task.

7.2 Frobenius method

If $x = x_0$ is a singular point of the differential equation (8), then there exists <u>at least one solution</u> of the form

$$y(x) = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r}$$

where the number r and the c_k's are constants to be determined. The series will converge at least on some interval $0 < x - x_0 < R$.

<u>An Introduction to the Method of Frobenius</u>

Before actually starting the method, there are two "pre-steps":

Pre-step 1: Choose a value for x_0. If conditions are given for $y(x)$ at some point, then use that point for x_0. Otherwise, choose x_0 as convenient — which usually choose $x_0 = 0$.

Pre-step 2: Get the differential equation into the form

$$A(x)y'' + B(x)y' + C(x)y = 0$$

where A, B, and C are polynomials.

Now for the basic method of Frobenius:

Step 1: (a) Start by assuming a solution of the form

$$y = y(x) = (x - x_0)^r \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

where a_0 is an arbitrary constant. Since it is arbitrary, we can and will assume $a_0 \neq 0$ in the following computations.

(b) Then simplify the formula for the following computations by bringing the $(x - x_0)^r$ factor into the summation,

$$y = y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^{k+r}$$

(c) And then compute the corresponding modified power series for y' and y'' from the assumed series for y by differentiating "term-by-term".

Step 2: Plug these series for y , y' , and y'' back into the differential equation, "multiply things out", and divide out the $(x - x_0)^r$ to get the left side of your equation in the form of the sum of a few power series.

Some Notes:

ii. Absorb any *x*'s in *A*, *B* and *C* (of the differential equation) into the series.

iii. Dividing out the $(x - x_0)^r$ isn't necessary, but it simplifies the expressions slightly and reduces the chances of silly errors later.

iv. You may want to turn your paper sideways for more room!

Step 3: For each series in your last equation, do a change of index so that each series looks like

$$\sum_{n=\text{something}}^{\infty} [\text{something not involving } x](x - x_0)^n$$

Be sure to appropriately adjust the lower limit in each series.

Step 4: Convert the sum of series in your last equation into one big series. The first few terms will probably have to be written separately. Simplify what can be simplified.

Observe that the end result of this step will be an equation of the form

some big power series = 0

This, in turn, tells that each term that big power series must be 0.

Step 5: The first term in the last equation just derived will be of the form

$$a_0[\text{formula of } r](x - x_0)^{\text{something}}$$

But, remember, each term in that series must be 0. So we must have

$$a_0[\text{formula of } r] = 0$$

Moreover, since $a_0 \neq 0$ (by assumption), the above must reduce to

$$\text{formula of } r = 0$$

This is the *indicial equation* for *r*. It will always be a quadratic equation for *r* (i.e., of the form $\alpha r^2 + \beta r + \delta = 0$). Solve this equation for *r*. You will get two solutions (sometimes called either the *exponents* of the solution or the *exponents* of the singularity). Denote them by r_2 and r_1 with $r_2 \leq r_1$

Step 6: Using r_1, the larger *r* just found:

(a) Plug r_1 into the last series equation (and simplify, if possible). This will give you an equation of the form

$$\sum_{n=n_0}^{\infty} [n^{th} \text{ formula of } a_k{}'s\ ](x - x_0)^n = 0$$

Since each term must vanish, we have

$$n^{th} \text{ formula of } a_k's = 0 \qquad\qquad \text{for } \ n_0 \leq n$$

(b) Solve this for

$$a_{\text{highest index}} = \text{formula of } n \text{ and lower indexed } a_k{}'s$$

A few of these equations may need to be treated separately, but you will also obtain a relatively simple formula that holds for all indices above some fixed value. This formula is the recursion formula for computing each coefficient a_n from the previously computed coefficients.

(c) To simplify things just a little, do another change of indices so that the recursion formula just derived is rewritten as

$$a_k = \text{formula of } k \text{ and lower indexed coefficients}$$

Step 7: Use the recursion formula (and any corresponding formulas for the lower-order terms) to find all the a_k's in terms of a_0 and, possibly, one other a_m. Look for patterns!

Step 8: Using $r = r_1$ along with the formulas just derived for the coefficients, write out the resulting series for *y*. Try to simplify it and factor out the arbitrary constant(s).

Step 9: If the indicial equation had two distinct solutions, now repeat steps 6 through 8 with the smaller *r*, r_2. Sometimes (but not always) this will give you a second independent solution to the differential equation. Sometimes, also, the series formula derived in this mega-step will include the series formula already derived.

Step 10: If the last step yielded y as an arbitrary linear combination of two different series, then that is the general solution to the original differential equation. If the last step yielded y as just one arbitrary constant times a series, then the general solution to the original differential equation is the linear combination of the two series obtained at the end of steps 8 and 9. Either way, write down the general solution (using different symbols for the two different arbitrary constants!). If step 9 did not yield a new series solution, then at least write down the one solution previously derived, noting that a second solution is still needed for the general solution to the differential equation.

Last Step: See if you recognize the series as the series for some well-known function (you probably won't!).

The following Bessel's equation of order ½ will be solved to illustrate the method.

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left[1 - \frac{1}{4x^2}\right]y = 0$$

Pre-step 1: There is no initial values at any point, so we will choose x_0 as simply as possibly; namely, $x_0 = 0$, which is a regular singular point.

Pre-step 2: To get the given differential equation into the form desired, we multiply the equation by $4x^2$. That gives us the differential equation

$$4x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + [4x^2 - 1]y = 0$$

Step 1: Since we've already decided $x_0 = 0$, we assume

$$y(x) = x^r\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+r}$$

with $a_0 \neq 0$. Differentiating this term-by-term, we see that

$$y' = \frac{d}{dx}\sum_{k=0}^{\infty} a_k x^{k+r} = \sum_{k=0}^{\infty}\frac{d}{dx}[a_k x^{k+r}] = \sum_{k=0}^{\infty}(k+r)a_k x^{k+r-1}$$

$$y'' = \frac{d}{dx}\sum_{k=0}^{\infty}(k+r)a_k x^{k+r-1} = \sum_{k=0}^{\infty}\frac{d}{dx}[(k+r)a_k x^{k+r-1}] = \sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^{k+r-2}$$

Step 2: Combining the above series formulas for y, y' and y'' with our differential equation, we get

$$0 = 4x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + [4x^2 - 1]y$$

$$= 4x^2 \sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^{k+r-2} + 4x \sum_{k=0}^{\infty}(k+r)a_k x^{k+r-1} + [4x^2 - 1] \sum_{k=0}^{\infty} a_k x^{k+r}$$

$$= 4x^2 \sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^{k+r-2} + 4x \sum_{k=0}^{\infty}(k+r)a_k x^{k+r-1} + 4x^2 \sum_{k=0}^{\infty} a_k x^{k+r} - 1 \sum_{k=0}^{\infty} a_k x^{k+r}$$

$$= \sum_{k=0}^{\infty}(k+r)(k+r-1)4a_k x^{k+r} + \sum_{k=0}^{\infty}(k+r)4a_k x^{k+r} + \sum_{k=0}^{\infty} 4a_k x^{k+2+r} + \sum_{k=0}^{\infty}(-1)a_k x^{k+r}$$

Dividing out the x^r from each term then yields

$$= \sum_{k=0}^{\infty}(k+r)(k+r-1)4a_k x^{k} + \sum_{k=0}^{\infty}(k+r)4a_k x^{k} + \sum_{k=0}^{\infty} 4a_k x^{k+2} + \sum_{k=0}^{\infty}(-1)a_k x^{k}$$

Step 3: In all but the third series, the "change of index" is trivial, $n = k$. In the third series, we will set $n = k + 2$ (equivalently, $n - 2 = k$). This means, in the third series, replacing k with $n - 2$, and replacing $k = 0$ with $n = 0 + 2 = 2$:

$$0 = \sum_{k=0}^{\infty}(k+r)(k+r-1)4a_k x^{k} + \sum_{k=0}^{\infty}(k+r)4a_k x^{k} + \sum_{k=0}^{\infty} 4a_k x^{k+2} + \sum_{k=0}^{\infty}(-1)a_k x^{k}$$

$n = k$	$n = k$	$n = k+2$	$n = k$

$$= \sum_{n=0}^{\infty}(n+r)(n+r-1)4a_n x^{n} + \sum_{n=0}^{\infty}(n+r)4a_n x^{n} + \sum_{n=2}^{\infty} 4a_{n-2} x^{n} + \sum_{n=0}^{\infty}(-1)a_n x^{n}$$

Step 4: Since one of the series in the last equation begins with $n = 2$, we need to separate out the terms corresponding to $n = 0$ and $n = 1$ in the other series before combining series:

$$0 = 4a_0(0+r)(0+r-1)x^0 + 4a_1(1+r)(1+r-1)x^1 + \sum_{n=2}^{\infty}(n+r)(n+r-1)4a_n x^n + 4a_0(0+r)x^0 + 4a_1(1+r)x^1 + \sum_{n=2}^{\infty}(n+r)4a_n x^n + \sum_{n=2}^{\infty} 4a_{n-2}x^n - a_0 x^0 - a_1 x^1 + \sum_{n=2}^{\infty}(-1)a_n x^n$$

So our differential equation reduces to

$$a_0[4r^2 - 1]x^0 + a_1[4r^2 + 8r + 3]x^1 + \sum_{n=2}^{\infty} a_n[4(n+r)^2 - 1) + 4a_{n-2}]x^n = 0 \qquad (*)$$

Observe that the end result of this step will be an equation of the form

some big power series = 0

This, in turn, tells us that each term that big power series must be 0.

Step 5: The first term in the "big series" is the first term in the equation

$$a_0[4r^2 - 1]x^0$$

Since this must be zero (and $a_0 \neq 0$ by assumption) the indicial equation is

$$4r^2 - 1 = 0$$

Thus,

$$r = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$$

Following the convention given,

$$r_2 = -\frac{1}{2} \qquad \text{and} \qquad r_1 = \frac{1}{2}$$

Step 6: Letting $r = r_1 = \frac{1}{2}$, equation (*) yields

$$a_0\left[4\left(\frac{1}{2}\right)^2 - 1\right]x^0 + a_1\left[4\left(\frac{1}{2}\right)^2 + 8\left(\frac{1}{2}\right) + 3\right]x^1 + \sum_{n=2}^{\infty} a_n\left[4\left(n + \frac{1}{2}\right)^2 - 1) + 4a_{n-2}\right]x^n = 0$$

$$a_0 0x^0 + a_1 8x^1 + \sum_{n=2}^{\infty}[a_n(4n^2 + 4n + 1 - 1) + 4a_{n-2}]x^n = 0$$

The first term vanishes (as it should since $r = 1/2$ satisfies the indicial equation, which came from making the first term vanish). Doing a little more simple algebra, we see that, with r = 1/2 , equation (*) reduces to

$$0a_0x^0 + 8a_1x^1 + \sum_{n=2}^{\infty} 4[n(n+1)a_n + a_{n-2}]x^n = 0 \qquad (**)$$

From the above series, we must have

$$n(n+1)a_n + a_{n-2} = 0 \qquad \text{for} \qquad n = 2, 3, 4, \ldots$$

Solving for a_n leads to the recursion formula

$$a_n = \frac{-1}{n(n+1)}a_{n-2} \qquad \text{for} \qquad n = 2, 3, 4, \ldots$$

Using the trivial change of index, $k = n$, this is

$$a_k = \frac{-1}{k(k+1)} a_{k-2} \qquad \text{for} \qquad k = 2, 3, 4, \ldots$$

Step 7: From the first two terms in equation (**),

$$0a_0 = 0 \qquad \Longrightarrow \qquad a_0 \text{ is arbitrary}$$

$$8a_1 = 0 \qquad \Longrightarrow \qquad a_1 = 0$$

Using these values and the recursion formula with $k = 2, 3, 4, \ldots$ (and looking for patterns):

$$a_2 = \frac{-1}{2(2+1)} a_{2-2} = \frac{-1}{2 \cdot 3} a_0$$

$$a_3 = \frac{-1}{3(3+1)} a_{3-2} = \frac{-1}{3 \cdot 4} a_1 = \frac{-1}{3 \cdot 4} \cdot 0 = 0$$

$$a_4 = \frac{-1}{4(4+1)} a_{4-2} = \frac{-1}{4 \cdot 5} a_2 = \frac{-1}{4 \cdot 5} \cdot \frac{-1}{2 \cdot 3} a_0 = \frac{(-1)^2}{5 \cdot 4 \cdot 3 \cdot 2} a_0 = \frac{(-1)^2}{5!} a_0$$

$$a_5 = \frac{-1}{5(5+1)} a_{5-2} = \frac{-1}{5 \cdot 6} a_3 = \frac{-1}{5 \cdot 6} \cdot 0 = 0$$

$$a_6 = \frac{-1}{6(6+1)} a_{6-2} = \frac{-1}{6 \cdot 7} a_4 = \frac{-1}{7 \cdot 6} \cdot \frac{(-1)^2}{5!} a_0 = \frac{(-1)^3}{7!} a_0$$

$$\vdots$$

The patterns should be obvious here:

$$a_k = 0 \qquad \text{for} \qquad k = 1, 3, 5, 7, \ldots$$

and

$$a_k = \frac{(-1)^{k/2}}{(k+1)!} a_0 \qquad \text{for} \qquad k = 2, 4, 6, 8, \ldots$$

Using $k = 2m$, this be written more conveniently as

$$a_{2m} = (-1)^m \frac{a_0}{(2m+1)!} \qquad \text{for} \qquad m = 1, 2, 3, 4, 5, \ldots$$

Step 8: Plugging $r = 1/2$ and the formulas just derived for the a_n's into the formula originally assumed for y, we get

$$y = x^r \sum_{k=0}^{\infty} a_k x^k$$

$$= x^r \left[\sum_{\substack{k=0 \\ k\ odd}}^{\infty} a_k x^k + \sum_{\substack{k=0 \\ k\ even}}^{\infty} a_k x^k \right]$$

$$= x^{1/2} \left[\sum_{\substack{k=0 \\ k\ odd}}^{\infty} 0 \cdot x^k + \sum_{m=0}^{\infty} (-1)^m \frac{a_0}{(2m+1)!} x^{2m} \right]$$

$$= x^{1/2} \left[0 + a_0 \sum_{m=0}^{\infty} (-1)^m \frac{1}{(2m+1)!} x^{2m} \right]$$

So one solution to Bessel's equation of order 1/2 is given by

$$y = a_0 x^{1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m}$$

Step 9: Letting $r = r_2 = -\frac{1}{2}$ in equation (*) yields

$$a_0 \left[4\left(-\frac{1}{2}\right)^2 - 1 \right] x^0 + a_1 \left[4\left(-\frac{1}{2}\right)^2 + 8\left(-\frac{1}{2}\right) + 3 \right] x^1 + \sum_{n=2}^{\infty} a_n \left[4\left(n - \frac{1}{2}\right)^2 - 1) + 4a_{n-2} \right] x^n = 0$$

$$a_0 0 x^0 + a_1 0 x^1 + \sum_{n=2}^{\infty} [a_n(4n^2 - 4n + 1 - 1) + 4a_{n-2}] x^n = 0$$

Then …

⋮ {"Fill in the dots" in the last statement. That is, do all the computations that were omitted.}

yielding

$$y = a_0 x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m} + a_1 x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}$$

Note that the second series term is the same series (slightly rewritten) since $x^{-1/2}x^{2m+1} = x^{1/2}x^{2m}$

Step 10: We are in luck. In the last step we obtained *y* as the linear combination of two different series. So

$$y = a_0 x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m} + a_1 x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}$$

is the general solution to the original differential equation — Bessel's equation of order 1/2.

Last Step: Our luck continues! The two series are easily recognized as the series for the sine and the cosine functions:

$$y = a_0 x^{-1/2} \cos x + a_1 x^{-1/2} \sin x$$

So the general solution to Bessel's equation of order 1/2 is

$$= a_0 \frac{\cos x}{\sqrt{x}} + a_1 \frac{\sin x}{\sqrt{x}}$$

<u>Advice and Comments</u>

1. If you get something like

$$2a_1 = 0$$

then you know $a_1 = 0$. On the other hand, if you get something like

$$0a_1 = 0$$

then you have an equation that tells you nothing about a_1 . This means that a_1 is an arbitrary constant (unless something else tells you otherwise).

2. If the recursion formula blows up at some point, then some of the coefficients must be zero. For example, if

$$a_n = \frac{3}{(n+2)(n-5)} a_{n-2}$$

then, for $n = 5$,

$$a_n = \frac{3}{(7)(0)} a_3 = \infty \cdot a_3$$

which can only make sense if $a_3 = 0$. Note also, that, unless otherwise indicated, a_5 here would be arbitrary. (Remember, the last equation is equivalent to $(7)(0)a_5 = 3a_3$.)

3 If you get a coefficient being zero, it is a good idea to check back using the recursion formula to see if any of the previous coefficients must also be zero, or if many of the following coefficients are zero. In some cases, you may find that an "infinite" series solution only contains a finite number of nonzero terms, in which case we have a "terminating series"; i.e., a solution which is simply a polynomial.

On the other hand, obtaining $a_0 = 0$, contrary to our basic assumption that $a_0 \neq 0$, tells you that there is no series solution of the form assumed for the basic Frobenius method using that value of *r*.

4. It is possible to end up with a three term recursion formula, say,

$$a_n = \frac{1}{n^2+1} a_{n-1} + \frac{2}{3n(n+3)} a_{n-2}$$

This, naturally, makes "finding patterns" rather difficult.

5. Keep in mind that, even if you find that "finding patterns" and describing them by "nice" formulas is beyond you, you can always use the recursion formulas to compute (or have a computer compute) as many terms as you wish of the series solutions.

6. The computations can become especially messy and confusing when $a_0 \neq 0$. In this case, simplify matters by using the substitutions

$$Y(X) = y(x) \qquad \text{with} \qquad X = x - x_0$$

You can then easily verify that, under these substitutions,

$$Y'^{(X)} = \frac{dY}{dX} = \frac{dy}{dx} = y(x)$$

and the differential equation

$$A(x)\frac{d^2y}{dx^2} + B(x)\frac{dy}{dx} + C(x)y = 0$$

becomes

$$A_1(X)\frac{d^2Y}{dX^2} + B_1(X)\frac{dY}{dX} + C_1(X)Y = 0$$

with

$$A_1(X) = A(X + x_0)\,, \quad B_1(X) = B(X + x_0) \quad \text{and} \quad C_1(X) = C(X + x_0)$$

Use the method of Frobenius to find the modified power series solutions

$$Y(X) = (X)^r \sum_{k=0}^{\infty} a_k X^k$$

for equation $A_1(X)\frac{d^2Y}{dX^2} + B_1(X)\frac{dY}{dX} + C_1(X)Y = 0$. The corresponding solutions to the original differential equation, equation $A(x)\frac{d^2y}{dx^2} + B(x)\frac{dy}{dx} + C(x)y = 0$, are then given from this via the above substitution,

$$y(x) = Y(X) = (X)^r \sum_{k=0}^{\infty} a_k X^k = (x - x_0)^r \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

<u>The Big Theorem on the Frobenius Method</u>

Let x_0 be a regular singular point (on the real line) for

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = 0$$

Then the indicial equation arising in the basic method of Frobenius exists and is a quadratic equation with two solutions r_1 and r_2 (which may be one solution, repeated). If r_2 and r_1 are real, assume $r_2 \leq r_1$. Then:

1. The basic method of Frobenius will yield at least one solution of the form

$$y_1(x) = (x - x_0)^{r_1} \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

where a_0 is the one and only arbitrary constant.

2. If $r_1 - r_2$ is not an integer, then the basic method of Frobenius **will** yield a second independent solution of the form

$$y_2(x) = (x - x_0)^{r_2} \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

where a_0 is an arbitrary constant.

3. If $r_1 - r_2 = N$ is positive integer, then the method of Frobenius **might** yield a second independent solution of the form

$$y_2(x) = (x - x_0)^{r_2} \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

where a_0 is an arbitrary constant. If it doesn't, then a second independent solution exists of the form

$$y_2(x) = y_1(x) \ln|x - x_0| + (x - x_0)^{r_2} \sum_{k=0}^{\infty} b_k (x - x_0)^k$$

or, equivalently

$$y_2(x) = y_1(x) \left[\ln|x - x_0| + (x - x_0)^{-N} \sum_{k=0}^{\infty} c_k (x - x_0)^k \right]$$

where b_0 and c_0 are nonzero constants.

4. If $r_1 = r_2$, then there is a second solution of the form

$$y_2(x) = y_1(x) \ln|x - x_0| + (x - x_0)^{1+r_1} \sum_{k=0}^{\infty} b_k (x - x_0)^k$$

or, equivalently

$$y_2(x) = y_1(x) \left[\ln|x - x_0| + (x - x_0)^{l} \sum_{k=0}^{\infty} c_k (x - x_0)^k \right]$$

In this case, b_0 and c_0 might be zero.

Moreover, if we let R be the distance between xo and the nearest singular point (other than x_o) in the complex plane (with $R = \infty$ if x_o is the only singular point), then the series solutions described above converge at least on the intervals ($x_o - R$, x_o) and (x_o, $x_o + R$).

Example 7.1: Apply the power series method to the following differential equations:

$$y'' + \frac{1}{2x} y' + \frac{1}{4x} y = 0$$

Answer:

Regular singular point at $x = 0$.

Multiply the equation with $4x$:

$$4xy'' + 2y' + y = 0$$

Assume solution:

$$y = \sum_{m=0}^{\infty} a_m \, x^{m+r}$$

Then,

$$y' = \sum_{m=0}^{\infty} (m + r) \, a_m x^{m+r-1}$$

$$y'' = \sum_{m=0}^{\infty} (m + r) \, (m + r - 1) a_m x^{m+r-2}$$

Hence,

$$4x \sum_{m=0}^{\infty} (m + r) \, (m + r - 1) a_m x^{m+r-2} + 2 \sum_{m=0}^{\infty} (m + r) \, a_m x^{m+r-1} + \sum_{m=0}^{\infty} a_m \, x^{m+r} = 0$$

$$4 \sum_{m=0}^{\infty} (m + r) \, (m + r - 1) a_m x^{m+r-1} + 2 \sum_{m=0}^{\infty} (m + r) \, a_m x^{m+r-1} + \sum_{m=0}^{\infty} a_m \, x^{m+r} = 0$$

For the first and second summation, let $k + r = m + r - 1$ which implies $m = k + 1$

For the third summation, let $k + r = m + r$, which implies $m = k$

$$4 \sum_{k=-1}^{\infty} (k + 1 + r) \, (k + r) a_{k+1} x^{k+r} + 2 \sum_{k=-1}^{\infty} (k + 1 + r) \, a_{k+1} x^{k+r} + \sum_{k=0}^{\infty} a_k \, x^{k+r} = 0$$

$$4(r)(r-1)a_0x^{r-1} + 4\sum_{k=0}^{\infty}(k+1+r)\,(k+r)a_{k+1}x^{k+r} + 2(r)a_0\,x^{r-1} + 2\sum_{k=0}^{\infty}(k+1+r)\,a_{k+1}x^{k+r}$$

$$+\sum_{k=0}^{\infty} a_k\,x^{k+r} = 0$$

$$[4r(r-1)+2r]a_0x^{r-1} + \left(\left[\sum_{k=0}^{\infty} 4(k+1+r)\,(k+r)a_{k+1} + 2(k+1+r)a_{k+1} + a_k\right]x^{k+r}\right) = 0$$

Indicial Equation:

$$4r(r-1)+2r = 0$$

$$\Rightarrow \quad 4r^2 - 4r + 2r = 0$$

$$\Rightarrow \quad r^2 - \frac{1}{2}r = 0$$

$$\Rightarrow \quad r\left(r-\frac{1}{2}\right) = 0$$

$$\therefore \quad r_2 = 0\,;\; r_1 = \tfrac{1}{2}$$

Distinct roots not differing by integer (including complex conjugates)

$$4(k+1+r)(k+r)a_{k+1} + 2(k+1+r)a_{k+1} + a_k = 0$$

$$a_{k+1} = \frac{-a_k}{4(k+1+r)(k+r)+2(k+1+r)} \qquad k = 0, 1, 2, \ldots$$

First solution: $r = \frac{1}{2}$

$$a_{k+1} = \frac{-a_k}{4\left(k+1+\frac{1}{2}\right)\left(k+\frac{1}{2}\right)+2\left(k+1+\frac{1}{2}\right)} \qquad k = 0, 1, 2, \ldots$$

$$a_1 = \frac{-a_0}{3\cdot 2} = -\frac{a_0}{3!}$$

$$a_2 = \frac{-a_1}{5\cdot 4} = -\frac{-a_0}{5\cdot 4\cdot 3!} = \frac{a}{5!}$$

$$a_3 = \frac{-a_2}{7\cdot 6} = -\frac{a_0}{7\cdot 6\cdot 5!} = -\frac{a_0}{7!}$$

$\vdots$

In general, let $a_0 = 1$

$$y_1(x) = x^{1/2}\left(1 - \frac{1}{6}x + \frac{1}{120}x^2 - \frac{1}{5040}x^3 \pm \cdots\right) = x^{1/2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}x^m$$

Second solution: $r = 0$

$$A_{k+1} = \frac{-A_k}{4(k+1)(k) + 2(k+1)} \qquad k = 0, 1, 2, \ldots$$

$$A_1 = \frac{-A_0}{2 \cdot 1} = -\frac{A_0}{2!}$$

$$A_2 = \frac{-A_1}{4 \cdot 3} = -\frac{-A_0}{4 \cdot 3 \cdot 2!} = \frac{A_0}{4!}$$

$$A_3 = \frac{-A_2}{6 \cdot 5} = -\frac{A_0}{6 \cdot 5 \cdot 4!} = -\frac{A_0}{6!}$$

$\vdots$

In general, let $A_0 = 1$

$$y_2(x) = x^0\left(1 - \frac{1}{2}x + \frac{1}{24}x^2 - \frac{1}{720}x^3 \pm \cdots\right) = \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}x^m$$

<u>Example 7.2:</u> Solve the following differential equations using power series method:

$$x(x-1)y'' + (3x-1)y' + y = 0$$

<u>Answer</u>:

Assume solution:

$$y = \sum_{m=0}^{\infty} a_m x^{m+r}$$

Hence,

$$x(x-1)\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-2} + (3x-1)\sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1} + \sum_{m=0}^{\infty}a_m x^{m+r} = 0$$

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r} - \sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-1} + 3\sum_{m=0}^{\infty}(m+r)a_m x^{m+r}$$
$$- \sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1} + \sum_{m=0}^{\infty}a_m x^{m+r} = 0$$

For the second and fourth summation, let *k* + *r* = *m* + *r* – 1 which implies *m* = *k* + 1

For the first, third, and fifth summation, let *k* + *r* = *m* + *r*, which implies *m* = *k*

$$\sum_{k=0}^{\infty}(k+r)\,(k+r-1)a_k x^{k+r} - \sum_{k=-1}^{\infty}(k+1+r)(k+r)\,a_{k+1}x^{k+r} + 3\sum_{k=0}^{\infty}(k+r)\,a_k x^{k+r} - \sum_{k=-1}^{\infty}(k+1+r)\,a_{k+1}x^{k+r} + \sum_{k=0}^{\infty}a_k\,x^{k+r} = 0$$

$$\sum_{k=0}^{\infty}(k+r)\,(k+r-1)a_k x^{k+r} - (r)(r-1)a_0 x^{r-1} - \sum_{k=0}^{\infty}(k+1+r)(k+r)\,a_{k+1}x^{k+r} + 3\sum_{k=0}^{\infty}(k+r)\,a_k x^{k+r} - (r)a_0 x^{r-1} - \sum_{k=0}^{\infty}(k+1+r)\,a_{k+1}x^{k+r} + \sum_{k=0}^{\infty}a_k\,x^{k+r} = 0$$

$$(-(r)(r-1)-r)a_0 x^{r-1} + \left[\sum_{k=0}^{\infty}(k+r)\,(k+r-1)a_k - (k+1+r)(k+r)a_{k+1} + 3(k+r)a_k - (k+1+r)a_{k+1} + a_k\right]x^{k+r} = 0$$

Indicial Equation:

$-(r)(r-1) - r = 0$

$\Rightarrow \qquad -r^2 + r - r = 0$

$\Rightarrow \qquad r^2 = 0$

$\therefore \qquad r_1 = 0\,;\ r_2 = 0$ **Double root**

$$(k+r)(k+r-1)a_k - (k+1+r)(k+r)a_{k+1} + 3(k+r)a_k - (k+1+r)a_{k+1} + a_k = 0$$

$$a_{k+1} = a_k \qquad k = 0, 1, 2, \ldots$$

First solution: *r* = 0

$a_{k+1} = a_k \qquad k = 0, 1, 2, \ldots$

$a_1 = a_0$

$a_2 = a_1 = a_0$

$a_3 = a_2 = a_0$

$\vdots$

In general, let $a_0 = 1$

$$y_1(x) = x^0(1 + x + x^2 + x^3 + \cdots) = \sum_{m=0}^{\infty} x^m = \frac{1}{1-x}$$

Second solution: $r = 0$

$A_{k+1} = A_k \qquad k = 0, 1, 2, \ldots$

$A_1 = A_0$

$A_2 = A_1 = A_0$

$A_3 = A_2 = A_0$

$\vdots$

In general, let $A_0 = 1$

$$y_2(x) = y_1(x)\ln x + x^1\left(\sum_{m=0}^{\infty} x^m\right) = \frac{1}{1-x}\ln x + x^1\left(\frac{1}{1-x}\right)$$

$$= \frac{1}{1-x}(\ln x - x)$$

Example 7.3: Solve the following differential equations using power series method:

$$(x^2 - 1)x^2 y'' - (x^2 + 1)xy' + (x^2 + 1)y = 0$$

Answer:

Assume solution:

$$y = \sum_{m=0}^{\infty} a_m x^{m+r}$$

Hence,

$$(x^2 - 1)x^2 \sum_{m=0}^{\infty} (m + r)(m + r - 1)a_m x^{m+r-2} - (x^2 + 1)x \sum_{m=0}^{\infty} (m + r)\, a_m x^{m+r-1} + (x^2 + 1) + \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

$$\sum_{m=0}^{\infty}(m+r)\,(m+r-1)a_m x^{m+r+2} - \sum_{m=0}^{\infty}(m+r)\,(m+r-1)a_m x^{m+r} - \sum_{m=0}^{\infty}(m+r)\,a_m x^{m+r+2}$$
$$+ \sum_{m=0}^{\infty}(m+r)\,a_m x^{m+r} + \sum_{m=0}^{\infty} a_m\, x^{m+r+2} + \sum_{m=0}^{\infty} a_m\, x^{m+r} = 0$$

$$\sum_{m=0}^{\infty}(m+r-1)\,a_m x^{m+r+2} - \sum_{m=0}^{\infty}(m+r-1)(m+r+1)\,a_m x^{m+r} = 0$$

For the first summation, let *k* + *r* = *m* + *r* +2 which implies *m* = *k* - 2

For the second summation, let *k* + *r* = *m* + *r*, which implies *m* = *k*

$$\sum_{k=2}^{\infty}(k+r-3)^2\,a_{k-2}x^{k+r} - \sum_{k=0}^{\infty}(k+r-1)(k+r+1)\,a_k x^{k+r} = 0$$

$$\sum_{k=2}^{\infty}[(k+r-3)^2 a_{k-2} - (k+r-1)(k+r+1)a_k]\,x^{k+r} - (r-1)(r+1)a_0 x^r - (r)(r+2)a_1 x^{r+1}$$
$$= 0$$

Indicial Equation:

$-(r-1)(r+1) = 0$

$\therefore \qquad r_1 = 1\,;\; r_2 = -1$ **Roots differing by an integer**

$(k+r-3)^2 a_{k-2} - (k+r-1)(k+r+1)a_k = 0$

$(k+r-1)(k+r+1)a_k = 0$

$$a_k = \frac{(k+r-3)^2 a_{k-2}}{(k+r-1)(k+r+1)} \qquad k = 2, 3, 4, \ldots$$

$-(r)(r+2)a_1 = 0$

$(-r^2 - 2r)a_1 = 0$

$a_1 = 0 \qquad$ since $(-r^2 - 2r) \neq 0$

First solution: *r* = 1

$$a_k = \frac{(k-2)^2 a_{k-2}}{(k)(k+2)} \qquad k = 2, 3, 4, \dots$$

$a_1 = 0$

$a_2 = 0$

$a_3 = 0$

$\vdots$

In general,

$y_1(x) = x^1(a_0 + a_1x + a_2x^2 + \cdots) = a_0x$

Second solution: r = -1

$$A_k = \frac{(k-4)^2 A_{k-2}}{(k-2)(k)} \qquad k = 2, 3, 4, \dots$$

When k = 2, there is no solution. Hence, there will be no solution in the form of power series and reduction of order should be used for getting the second solution.

Let $y_2(x) = xu(x)$ be a solution to the differential equation

Then $y_2'(x) = xu'(x) + u(x)$

And $y_2''(x) = xu''(x) + u'(x) + u'(x) = xu''(x) + 2u'(x)$

Substitute into the equation,

$$(x^2-1)x^2[xu''(x) + 2u'(x)] - (x^2+1)x[xu'(x) + u(x)] + (x^2+1)[xu(x)] = 0$$

$$x^2(x^3 - x)u'' + x^2(x^2-3)u' = 0$$

$$(x^3 - x)u'' + (x^2-3)u' = 0$$

$$\frac{u''}{u'} = \frac{(x^2-3)}{(x^3-x)} = -\frac{3}{x} + \frac{1}{x+1} + \frac{1}{x-1}$$

$$\int \frac{u''}{u'}\, du = \int -\frac{3}{x}\, dx + \int \frac{1}{x+1}\, dx + \int \frac{1}{x-1} dx$$

$$\ln u' = -3\ln x + \ln(x+1) + \ln(x-1) = \ln\frac{(x+1)(x-1)}{x^3} = \ln\frac{x^2-1}{x^3}$$

$$u = \ln x + \frac{1}{2x^2}$$

Therefore, the second solution is

$$y_2(x) = xu(x) = x\left(\ln x + \frac{1}{2x^2}\right) = x\ln x + \frac{1}{2x}$$